

ON RS -COMPACT SPACES

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1. Introduction

In 1969, Singal and Mathur [16] defined a topological space X to be nearly-compact if every regular open cover of X has a finite subcover. In 1976, Thompson [20] defined a topological space X to be S -closed if every regular closed cover of X has a finite subcover. Recently, Hong [6] has introduced a new class of topological spaces called RS -compact spaces and characterized by the following property: "Every regular closed cover of a topological space X has a finite subfamily, the interiors of whose members cover X " (Theorem 3.5 below). It is obvious that RS -compactness implies S -closedness and near-compactness. It is known in [6] that these three concepts are equivalent to that of quasi H -closedness due to Porter and Thomas [14] if the space is extremally disconnected. It will be shown that RS -compactness neither implies compactness nor is implied by compactness even though the space is Hausdorff (Example 2.5 below).

The purpose of the present paper is to obtain several further properties of RS -compact spaces. §3 is concerned with characterizations of RS -compact spaces. In §4, we introduce subsets called RS -compact relative to a topological space. The class of such subsets is not necessarily contained in the class of RS -compact subspaces and also there exists a RS -compact subspace without being RS -compact relative to the space. By making use of such subsets we obtain some characterizations of RS -compact spaces. In §5, we investigate the conditions under which RS -compact subspaces are equivalent to subsets RS -compact relative to the space. The last section is devoted to the investigation of the preservation (inverse-preservation) theorems of RS -compact spaces.

2. Preliminaries

Throughout the present paper X means a topological space on which no separation axioms are assumed unless explicitly stated. Let S be a subset of a space X . The closure of S in X and the interior of S in X are denoted by $\text{Cl}_X(S)$ and $\text{Int}_X(S)$ (briefly, $\text{Cl}(S)$ and $\text{Int}(S)$), respectively. A set S is said to be *regular open* (resp. *regular closed*) if $\text{Int}(\text{Cl}(S))=S$ (resp. $\text{Cl}(\text{Int}(S))=S$). A set S is said to be *regular semi-open* [1] (resp. *semi-open* [8]) if there exists a regular open (resp. open) set O such that $O \subset S \subset \text{Cl}(O)$. It should be noticed that the complement of a regular semi-open set is also regular semi-open. The family of all regular semi-open (resp. regular open, regular closed) sets in X is denoted by $RSO(X)$ (resp. $RO(X)$, $RC(X)$). A space X is said to be *extremally disconnected* if for every open set O of X , $\text{Cl}(O)$ is open in X . The notation βN denotes the Stone-Ćech compactification of N .

DEFINITION 2.1. A space X is said to be *RS-compact* [6] (resp. *S-closed* [20]) if for every regular semi-open (resp. semi-open) cover $\{V_\alpha | \alpha \in \mathcal{V}\}$ of X , there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that

$$X = \bigcup \{\text{Int}(V_\alpha) | \alpha \in \mathcal{V}_0\} \quad (\text{resp. } X = \bigcup \{\text{Cl}(V_\alpha) | \alpha \in \mathcal{V}_0\}).$$

DEFINITION 2.2. A space X is said to be *nearly-compact* [16] (resp. *quasi H-closed* [14]) if for every open cover $\{V_\alpha | \alpha \in \mathcal{V}\}$ of X , there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that

$$X = \bigcup \{\text{Int}(\text{Cl}(V_\alpha)) | \alpha \in \mathcal{V}_0\} \quad (\text{resp. } X = \bigcup \{\text{Cl}(V_\alpha) | \alpha \in \mathcal{V}_0\}).$$

THEOREM 2.3 (Hong [6]). *For a space the following implications hold. If the space is extremally disconnected, then these four properties are equivalent.*

$$\begin{array}{ccc} RS\text{-compactness} & \Rightarrow & S\text{-closedness} \Rightarrow \text{quasi } H\text{-closedness} \\ & \Downarrow & \Uparrow \\ & \text{near-compactness} & \end{array}$$

REMARK 2.4. *RS-compactness and compactness are independent of each other even though the space is Hausdorff.*

EXAMPLE 2.5. Since βN is compact Hausdorff, the product space βN

$\times \beta N$ is compact Hausdorff. However, $\beta N \times \beta N$ is not S -closed [9, p. 193] and hence not RS -compact by Theorem 2.3. Therefore, there exists a compact Hausdorff space which is not RS -compact. Since an S -closed Hausdorff space is extremally disconnected [20, Theorem 7], by Example 3.17 of [4] there exists an S -closed Hausdorff (and hence RS -compact Hausdorff) space which is not compact. It should be noticed that the statement in [5] "compactness and RS -compactness are equivalent in an extremally disconnected space" is partially false.

3. Characterizations of RS -compact spaces

DEFINITION 3.1. A filter base $\{F_\alpha\}$ on X RS -converges to $x \in X$ if for each $V \in RSO(X)$ containing x there exists an F_α such that $F_\alpha \subset \text{Int}(V)$.

DEFINITION 3.2. A filter base $\{F_\alpha\}$ on X RS -accumulates at $x \in X$ if $F_\alpha \cap \text{Int}(V) \neq \emptyset$ for each F_α and each $V \in RSO(X)$ containing x .

LEMMA 3.3. Let \mathcal{F} be a maximal filter base on X . Then \mathcal{F} RS -accumulates at $x \in X$ if and only if \mathcal{F} RS -converges to $x \in X$.

The following theorems are obtained by the usual techniques and the proofs are thus omitted.

THEOREM 3.4. For a space X the following are equivalent:

- (1) X is RS -compact.
- (2) For each family $\{F_\alpha \mid \alpha \in \mathcal{V}\}$ of regular semi-open sets such that $\bigcap \{F_\alpha \mid \alpha \in \mathcal{V}\} = \emptyset$, there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $\bigcap \{\text{Cl}(F_\alpha) \mid \alpha \in \mathcal{V}_0\} = \emptyset$.
- (3) Each filter base RS -accumulates at some point of X .
- (4) Each maximal filter base RS -converges to some point of X .

THEOREM 3.5. For a space X the following are equivalent:

- (1) X is RS -compact.
- (2) For each regular closed cover $\{V_\alpha \mid \alpha \in \mathcal{V}\}$ of X , there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $X = \bigcup \{\text{Int}(V_\alpha) \mid \alpha \in \mathcal{V}_0\}$.
- (3) For each family $\{V_\alpha \mid \alpha \in \mathcal{V}\}$ of regular open sets of X such that $\bigcap \{V_\alpha \mid \alpha \in \mathcal{V}\} = \emptyset$, there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $\bigcap \{\text{Cl}(V_\alpha) \mid \alpha \in \mathcal{V}_0\} = \emptyset$.

4. Sets *RS*-compact relative to a space

In this section we introduce sets *RS*-compact relative to a space and obtain some characterizations of *RS*-compact spaces by making use of these sets. A subset S of X is said to be *RS*-compact if it is *RS*-compact as the subspace of X .

DEFINITION 4.1. A subset K of X is said to be

(1) *RS*-compact relative to X (resp. *S*-closed relative to X [9]) if for every cover $\{V_\alpha \mid \alpha \in \mathcal{V}\}$ of K by regular semi-open (resp. semi-open) sets of X , there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that

$$K \subset \bigcup \{\text{Int}(V_\alpha) \mid \alpha \in \mathcal{V}_0\} \quad (\text{resp. } K \subset \bigcup \{\text{Cl}(V_\alpha) \mid \alpha \in \mathcal{V}_0\});$$

(2) *N*-closed relative to X [2] (resp. quasi *H*-closed relative to X [14]) if for every cover $\{V_\alpha \mid \alpha \in \mathcal{V}\}$ of K by open sets of X , there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that

$$K \subset \bigcup \{\text{Int}(\text{Cl}(V_\alpha)) \mid \alpha \in \mathcal{V}_0\} \quad (\text{resp. } K \subset \bigcup \{\text{Cl}(V_\alpha) \mid \alpha \in \mathcal{V}_0\}).$$

LEMMA 4.2. Let X be extremally disconnected. Then, for a subset A of X the following are equivalent:

- (1) A is *RS*-compact relative to X .
- (2) A is *S*-closed relative to X .
- (3) A is *N*-closed relative to X .
- (4) A is quasi *H*-closed relative to X .

Proof. The proof is obvious. However, it should be noticed that the implications (1) \Rightarrow (2) \Rightarrow (4) and (1) \Rightarrow (3) \Rightarrow (4) hold without the assumption “extremally disconnected” on X .

REMARK 4.3. Two notions “*RS*-compact” and “*RS*-compact relative to X ” on a subset of X are independent of each other even though the subset is closed in X . However, it will be shown that on a regular open set of X these two notions are equivalent (see Theorem 5.7 below).

EXAMPLE 4.4. Since $\beta N - N$ is closed in the compact space βN , $\beta N - N$ is compact and hence *N*-closed relative to βN . Moreover, since βN is extremally disconnected, by Lemma 4.2 $\beta N - N$ is *RS*-compact relative to βN . However, $\beta N - N$ is not *S*-closed [9, Example 1.7] and hence not *RS*-compact by Theorem 2.3.

EXAMPLE 4.5. In [1, Example 1], $Y = \beta N_1 \cup \beta N_2$ is not S -closed, βN_1 and βN_2 are closed RS-compact subspaces of Y . The finite union of sets S -closed relative to Y is S -closed relative to Y . Therefore, βN_1 and βN_2 are not S -closed relative to Y and hence not RS-compact relative to Y by the proof of Lemma 4.2.

THEOREM 4.6. *For a nonempty subset A of X , the following are equivalent:*

- (1) A is RS-compact relative to X .
- (2) Every maximal filter base on X which meets A RS-converges to some $x \in A$.
- (3) Every filter base on X which meets A RS-accumulates at some $x \in A$.
- (4) For every cover $\{V_\alpha | \alpha \in \mathcal{V}\}$ of A by regular closed sets of X , there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $A \subset \cup \{\text{Int}(V_\alpha) | \alpha \in \mathcal{V}_0\}$.
- (5) For every family $\{V_\alpha | \alpha \in \mathcal{V}\}$ of regular open sets of X such that $[\cap \{V_\alpha | \alpha \in \mathcal{V}\}] \cap A = \phi$, there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $[\cap \{\text{Cl}(V_\alpha) | \alpha \in \mathcal{V}_0\}] \cap A = \phi$.
- (6) For every family $\{V_\alpha | \alpha \in \mathcal{V}\}$ of regular semi-open sets of X such that $[\cap \{V_\alpha | \alpha \in \mathcal{V}\}] \cap A = \phi$, there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $[\cap \{\text{Cl}(V_\alpha) | \alpha \in \mathcal{V}_0\}] \cap A = \phi$.

Proof. (1) \Rightarrow (2): Let \mathcal{F} be a maximal filter base on X which meets A . Assume that \mathcal{F} does not RS-converge to any $x \in A$. Then, since \mathcal{F} is a maximal filter base, by Lemma 3.3 \mathcal{F} does not RS-accumulate at any $x \in A$. For each $x \in A$ there exist $F_x \in \mathcal{F}$ and $V_x \in \text{RSO}(X)$ containing x such that $F_x \cap \text{Int}(V_x) = \phi$. The family $\{V_x | x \in A\}$ is a cover of A by regular semi-open sets of X . Thus, there exists a finite number of points x_1, x_2, \dots, x_n in A such that $A \subset \cup \{\text{Int}(V_{x_j}) | 1 \leq j \leq n\}$. Since \mathcal{F} is a filter base, there exists an $F_0 \in \mathcal{F}$ such that $F_0 \subset \cap \{F_{x_j} | 1 \leq j \leq n\}$. Therefore, we have $F_0 \cap A = \phi$. This contradicts that \mathcal{F} meets A .

(2) \Rightarrow (3): Let \mathcal{F} be a filter base on X which meets A . Then there exists a maximal filter base \mathcal{F}_0 on X such that $\mathcal{F} \subset \mathcal{F}_0$ and \mathcal{F}_0 meets A . By the hypothesis, \mathcal{F}_0 RS-converges to some $x \in A$. For any $F \in \mathcal{F}$ and any $V \in \text{RSO}(X)$ containing x , there exists an $F_0 \in \mathcal{F}_0$ such that $F_0 \subset \text{Int}(V)$ and hence we have $F \cap \text{Int}(V) \neq \phi$. This shows that \mathcal{F} RS-accumulates at $x \in A$.

(3) \Rightarrow (4): Let $\{V_\alpha | \alpha \in \mathcal{F}\}$ be a cover of A by regular closed sets of X . Let $\Gamma(\mathcal{F})$ denote the family of all finite subsets of \mathcal{F} . Assume that $A \not\subset \bigcup \{\text{Int}(V_\alpha) | \alpha \in \Delta\}$ for each $\Delta \in \Gamma(\mathcal{F})$. Then

$$\mathcal{F} = \{A \cap [\bigcap_{\alpha \in \Delta} (X - \text{Int}(V_\alpha))] | \Delta \in \Gamma(\mathcal{F})\}$$

is a filter base on X which meets A . By the hypothesis, \mathcal{F} *RS*-accumulates at some $x \in A$. However, for some $\alpha_0 \in \mathcal{F}$, $x \in V_{\alpha_0} \in \text{RSO}(X)$, $A \cap [X - \text{Int}(V_{\alpha_0})] \in \mathcal{F}$ and $\text{Int}(V_{\alpha_0}) \cap (A \cap [X - \text{Int}(V_{\alpha_0})]) = \emptyset$. This is a contradiction.

(4) \Rightarrow (5): Let $\{V_\alpha | \alpha \in \mathcal{F}\}$ be a family of regular open sets of X such that the intersection does not meet A . Then $\{X - V_\alpha | \alpha \in \mathcal{F}\}$ is a cover of A by regular closed sets of X . By the hypothesis, there exists a finite subset \mathcal{F}_0 of \mathcal{F} such that $A \subset \bigcup \{\text{Int}(X - V_\alpha) | \alpha \in \mathcal{F}_0\}$. Therefore, we obtain $A \cap [\bigcap \{\text{Cl}(V_\alpha) | \alpha \in \mathcal{F}_0\}] = \emptyset$.

(5) \Rightarrow (6): Let $\{V_\alpha | \alpha \in \mathcal{F}\}$ be a family of regular semi-open sets of X such that the intersection does not meet A . Since $V_\alpha \in \text{RSO}(X)$, we have $\text{Int}(V_\alpha) \in \text{RO}(X)$ and $\text{Cl}(V_\alpha) = \text{Cl}(\text{Int}(V_\alpha))$ for each $\alpha \in \mathcal{F}$. Therefore, by the hypothesis, we have $A \cap [\bigcap \{\text{Cl}(V_\alpha) | \alpha \in \mathcal{F}_0\}] = \emptyset$ for some finite subset \mathcal{F}_0 of \mathcal{F} .

(6) \Rightarrow (1): This follows from the fact that the complement of a regular semi-open set is also regular semi-open.

THEOREM 4.7. *If $A \in \text{RSO}(X)$ and B is *RS*-compact relative to X , then $A \cap B$ is *RS*-compact relative to X .*

Proof. Let $\{V_\alpha | \alpha \in \mathcal{F}\}$ be a cover of $A \cap B$ by regular semi-open sets of X . Then, we have $B \subset (X - A) \cup [\bigcup \{V_\alpha | \alpha \in \mathcal{F}\}]$ and $X - A \in \text{RSO}(X)$. Since B is *RS*-compact relative to X , there exists a finite subset \mathcal{F}_0 of \mathcal{F} such that

$$B \subset \text{Int}(X - A) \cup [\bigcup \{\text{Int}(V_\alpha) | \alpha \in \mathcal{F}_0\}].$$

Therefore, we obtain $A \cap B \subset \bigcup \{\text{Int}(V_\alpha) | \alpha \in \mathcal{F}_0\}$. This shows that $A \cap B$ is *RS*-compact relative to X .

The following corollaries are immediate consequences of Theorem 4.7.

COROLLARY 4.8. *If X is *RS*-compact and $A \in \text{RSO}(X)$, then A is *RS*-compact relative to X .*

COROLLARY 4.9. *Let X be RS -compact. If either $A \in RO(X)$ or $A \in RC(X)$, then A is RS -compact relative to X .*

REMARK 4.10. In an RS -compact space there exists a subset which is RS -compact relative to the space without being regular semi-open. Since βN is S -closed [20, Corollary] and extremally disconnected, by Theorem 2.3 βN is RS -compact. By Example 4.4, $\beta N - N$ is RS -compact relative to βN . However, since $Cl(N) = \beta N$, $\beta N - N \notin RSO(\beta N)$.

LEMMA 4.11. *Let X_0 be an open set of X . Then we have*

- (1) *If $A \subset X_0$, then $Int_X(A) = Int_{X_0}(A)$.*
- (2) *If $V \in RSO(X)$, then $V \cap X_0 \in RSO(X_0)$.*

Proof. (1) This is obvious.

(2) Let $V \in RSO(X)$, then there exists an $O \in RS(X)$ such that $O \subset V \subset Cl_X(O)$. Since X_0 is open in X , $O \cap X_0 \in RO(X_0)$ and

$$Cl_X(O) \cap X_0 \subset Cl_X(O \cap X_0) \cap X_0 = Cl_{X_0}(O \cap X_0).$$

This implies that $V \cap X_0 \in RSO(X_0)$.

THEOREM 4.12. *Let $X_0 \in RO(X)$ and $A \subset X_0$. Then A is RS -compact relative to X_0 if and only if A is RS -compact relative to X .*

Proof. Strong Necessity. Suppose that X_0 is open in X and A is RS -compact relative to X_0 . Let $\{V_\alpha \mid \alpha \in \mathcal{V}\}$ be any cover of A by regular semi-open sets of X . By Lemma 4.11, for each $\alpha \in \mathcal{V}$ $V_\alpha \cap X_0 \in RSO(X_0)$ and hence there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that

$$A \subset \bigcup \{Int_{X_0}(V_\alpha \cap X_0) \mid \alpha \in \mathcal{V}_0\}.$$

It follows from Lemma 4.11 that $Int_{X_0}(V_\alpha \cap X_0) = Int_X(V_\alpha \cap X_0) \subset Int_X(V_\alpha)$ for each $\alpha \in \mathcal{V}_0$. This implies that A is RS -compact relative to X .

Sufficiency. Suppose that $X_0 \in RO(X)$ and A is RS -compact relative to X . Let $\{V_\alpha \mid \alpha \in \mathcal{V}\}$ be a cover of A by regular semi-open sets of X_0 . For each $\alpha \in \mathcal{V}$, there exists $O_\alpha \in RO(X_0)$ such that $O_\alpha \subset V_\alpha \subset Cl_{X_0}(O_\alpha)$. Since $X_0 \in RO(X)$, for each $\alpha \in \mathcal{V}$ $O_\alpha \in RO(X)$ and $Cl_{X_0}(O_\alpha) \subset Cl_X(O_\alpha)$. Therefore, we have $V_\alpha \in RSO(X)$ for each $\alpha \in \mathcal{V}$ and hence for some finite subset \mathcal{V}_0 of \mathcal{V} $A \subset \bigcup \{Int_X(V_\alpha) \mid \alpha \in \mathcal{V}_0\}$. By Lemma 4.11, $Int_X(V_\alpha) = Int_{X_0}(V_\alpha)$ for each $\alpha \in \mathcal{V}_0$. This shows that A is RS -compact relative to X_0 .

A space X is said to be *weakly-Hausdorff* [18] if each point of X is the intersection of regular closed sets. A point x of X is said to be θ -adherent (resp. δ -adherent) point of a subset S of X if $\text{Cl}(V) \cap S \neq \phi$ (resp. $\text{Int}(\text{Cl}(V)) \cap S \neq \phi$) for every open neighborhood V of x in X . A subset S is said to be θ -closed (resp. δ -closed) [21] if S contains all θ -adherent (resp. δ -adherent) points of S .

THEOREM 4.13. *If X is weakly-Hausdorff and A is RS -compact relative to X , then A is θ -closed.*

Proof. Let x be a point of $X-A$. For each $a \in A$, there exist $V_a \in RO(X)$ and $U_a \in RC(X)$ such that $x \in V_a$, $a \in U_a$ and $V_a \cap U_a = \phi$. The family $\{U_a | a \in A\}$ is a cover of A by regular closed sets of X . By Theorem 4.6, there exists a finite number of points a_1, a_2, \dots, a_n in A such that $A \subset \cup \{\text{Int}(U_{a_j}) | 1 \leq j \leq n\}$. Since V_a and U_a are disjoint, we have $\text{Cl}(V_a) \cap \text{Int}(U_a) = \phi$ for each $a \in A$. Let $V_x = \cap \{V_{a_j} | 1 \leq j \leq n\}$, then $x \in V_x \in RO(X)$ and $A \cap \text{Cl}(V_x) = \phi$. This shows that A is θ -closed in X .

THEOREM 4.14. *If X is RS -compact and A is δ -closed in X , then A is RS -compact relative to X .*

Proof. Let $\{V_\alpha | \alpha \in \mathcal{V}\}$ be a cover of A by regular semi-open sets of X . For each point $x \in A$, there exists a $\bar{U}_x \in RO(X)$ such that $x \in \bar{U}_x$ and $U_x \cap A = \phi$. Since $\{U_x | x \in X-A\}$ is a cover of $X-A$ and $RO(X) \subset RSO(X)$, there exist a finite number of points x_1, x_2, \dots, x_n in $X-A$ and a finite subset \mathcal{V}_0 of \mathcal{V} such that

$$X = [\cup \{\text{Int}(V_\alpha) | \alpha \in \mathcal{V}_0\}] \cup [\cup \{U_{x_j} | 1 \leq j \leq n\}].$$

Therefore, we obtain $A \subset \cup \{\text{Int}(V_\alpha) | \alpha \in \mathcal{V}_0\}$. This shows that A is RS -compact relative to X .

COROLLARY 4.15. *Let X be S -closed weakly-Hausdorff. For a subset A of X the following are equivalent:*

- (1) A is θ -closed in X .
- (2) A is δ -closed in X .
- (3) A is RS -compact relative to X .

REMARK 4.16. Every S -closed weakly-Hausdorff space is extremally disconnected [4, Theorem 3.7]. Thus each property in 'Corollary 4.15

is equivalent to each one in Lemma 4.2.

We shall conclude this section by giving some characterizations of RS -compact spaces.

THEOREM 4.17. *For a space X the following are equivalent:*

- (1) X is RS -compact.
- (2) Every proper regular semi-open set is RS -compact relative to X .
- (3) Every proper regular closed set is RS -compact relative to X .
- (4) Every proper δ -closed set is RS -compact relative to X .

Proof. (1) \Rightarrow (2): This follows from Corollary 4.8.

(2) \Rightarrow (3): This is obvious since $RC(X) \subset RSO(X)$.

(3) \Rightarrow (1): Let $\{V_\alpha \mid \alpha \in \mathcal{F}\}$ be a regular semi-open cover of X . We may assume that V_{α_0} is proper for some $\alpha_0 \in \mathcal{F}$. There exists an $O \in RO(X)$ such that $O \subset V_{\alpha_0} \subset Cl(O)$. Then $X - O$ is a proper regular closed set of X and $X = Int(V_{\alpha_0}) \cup (X - O)$. Since $X - O \subset \cup \{V_\alpha \mid \alpha \in \mathcal{F}\}$, by the hypothesis there exists a finite subset \mathcal{F}_0 of \mathcal{F} such that $X - O \subset \cup \{Int(V_\alpha) \mid \alpha \in \mathcal{F}_0\}$. Therefore, we obtain $X = \cup \{Int(V_\alpha) \mid \alpha \in \mathcal{F}_0 \cup \{\alpha_0\}\}$. This shows that X is RS -compact.

(1) \Rightarrow (4): This follows from Theorem 4.14.

(4) \Rightarrow (3): This follows from the fact that every regular closed set is δ -closed.

5. RS -compact subspaces

A subset S of X is said to be RS -compact if the subspace S of X is RS -compact. As we have noticed in Remark 4.3, two notions “ RS -compact” and “ RS -compact relative to X ” on a subset of X are independent of each other. In this section, we shall investigate the conditions under which these two notions are equivalent.

REMARK 5.1 Not every subspace of a RS -compact space is RS -compact even though the subset is either open or closed in X . However, it will be shown that regular open subsets of RS -compact spaces are necessarily RS -compact (see Corollary 5.8 below)

EXAMPLE 5.2. By Remark 4.10, βN is RS -compact. Although N is

open in βN , neither N nor $\beta N - N$ is S -closed [9, Example 1.7]. Thus, by Theorem 2.3 neither N nor $\beta N - N$ is RS -compact.

LEMMA 5.3. *If X_0 is dense in X and $V_0 \in RSO(X_0)$, then $V_0 = V \cap X_0$ for some $V \in RSO(X)$.*

Proof. Since $V_0 \in RSO(X_0)$, there exists an $O_0 \in RO(X_0)$ such that $O_0 \subset V_0 \subset Cl_{X_0}(O_0)$. Moreover, there exists an $O \in RO(X)$ such that $O_0 = O \cap X_0$ [19, Lemma]. Put $V = V_0 \cup (Cl_X(O) - X_0)$. Then $V \cap X_0 = V_0$ and $O = O_0 \cup (O - X_0) \subset V$. Since $Cl_{X_0}(O_0) \subset Cl_X(O)$, we have $V \subset Cl_X(O)$ and hence $O \subset V \subset Cl_X(O)$. This shows that $V \in RSO(X)$.

THEOREM 5.4. *If A is dense in X and RS -compact relative to X , then A is RS -compact.*

Proof. Let $\{V_\alpha | \alpha \in \mathcal{V}\}$ be a cover of A by regular semi-open sets of the subspace A . By Lemma 5.3, for each $\alpha \in \mathcal{V}$ there exists a $U_\alpha \in RSO(X)$ such that $V_\alpha = U_\alpha \cap A$. Since $\{U_\alpha | \alpha \in \mathcal{V}\}$ is a cover of A by regular semi-open sets of X , there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $A \subset \cup \{Int_X(U_\alpha) | \alpha \in \mathcal{V}_0\}$. For each $\alpha \in \mathcal{V}_0$, we have

$$A \cap Int_X(U_\alpha) \subset Int_A(A \cap U_\alpha) = Int_A(V_\alpha).$$

Therefore, we obtain $A = \cup \{Int_A(V_\alpha) | \alpha \in \mathcal{V}_0\}$. This shows that A is RS -compact.

THEOREM 5.5. *If A is open RS -compact in X , then A is RS -compact relative to X .*

Proof. Let $\{V_\alpha | \alpha \in \mathcal{V}\}$ be a cover of A by regular semi-open sets of X . Then, by Lemma 4.11 $V_\alpha \cap A \in RSO(A)$ for each $\alpha \in \mathcal{V}$ and $A = \cup \{V_\alpha \cap A | \alpha \in \mathcal{V}\}$. There exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $A = \cup \{Int_A(V_\alpha \cap A) | \alpha \in \mathcal{V}_0\}$. Since A is open, by Lemma 4.11 we have $Int_A(V_\alpha \cap A) = Int_X(V_\alpha \cap A) \subset Int_X(V_\alpha)$ for each $\alpha \in \mathcal{V}_0$. Therefore we obtain $A \subset \cup \{Int_X(V_\alpha) | \alpha \in \mathcal{V}_0\}$. This shows that A is RS -compact relative to X .

COROLLARY 5.6. *Let A be an open dense subset of X . Then A is RS -compact if and only if it is RS -compact relative to X .*

Proof. This is an immediate consequence of Theorems 5.4 and 5.5.

THEOREM 5.7. *Let $A \in RO(X)$. Then A is RS -compact if and only if A is RS -compact relative to X .*

Proof. Strong Necessity. Suppose that A is an open RS -compact set of X . Then the proof follows from Theorem 5.5.

Sufficiency. Suppose that $A \in RO(X)$ and it is RS -compact relative to X . Let $\{V_\alpha | \alpha \in \mathcal{V}\}$ be a cover of A by regular semi-open sets of the subspace A . For each $\alpha \in \mathcal{V}$, there exists an $O_\alpha \in RO(A)$ such that $O_\alpha \subset V_\alpha \subset Cl_A(O_\alpha) \subset Cl_X(O_\alpha)$. Since $A \in RO(X)$, we have $O_\alpha \in RO(X)$ and hence $V_\alpha \in RSO(X)$ for each $\alpha \in \mathcal{V}$. There exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $A = \cup \{\text{Int}_X(V_\alpha) | \alpha \in \mathcal{V}_0\}$. Since $A \in RO(X)$, by Lemma 4.11 $\text{Int}_X(V_\alpha) = \text{Int}_A(V_\alpha)$ for each $\alpha \in \mathcal{V}_0$. This implies that A is RS -compact.

COROLLARY 5.8. *If X is RS -compact and $A \in RO(X)$, then A is RS -compact.*

Proof. This follows from Corollary 4.9 and Theorem 5.7.

A space X is said to be *submaximal* [1] if all dense subsets of X are open in X . In [1, Theorem 9], it is known that every quasi H -closed subspace of a submaximal extremally disconnected space is S -closed. We shall give an improvement of this result.

THEOREM 5.9. *Let X be a submaximal extremally disconnected space. If A is a quasi H -closed subspace, then it is RS -compact.*

Proof. Let $\{V_\alpha | \alpha \in \mathcal{V}\}$ be a cover of A by regular closed sets of the subspace A . Since every regular closed set is semi-open, by [13, Theorem 3.2] for each $\alpha \in \mathcal{V}$ there exists a semi-open set U_α of X such that $V_\alpha = U_\alpha \cap A$. By Theorem 8 of [1], U_α is open in X and hence V_α is open in the subspace A . Since A is quasi H -closed, there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $A = \cup \{Cl_A(V_\alpha) | \alpha \in \mathcal{V}_0\}$. Since $Cl_A(V_\alpha) = V_\alpha = \text{Int}_A(V_\alpha)$ for each $\alpha \in \mathcal{V}_0$, we have $A = \cup \{\text{Int}_A(V_\alpha) | \alpha \in \mathcal{V}_0\}$. It follows from Theorem 3.5 that A is RS -compact.

REMARK 5.10. In Theorem 5.9, the condition "submaximal" may not be dropped as the following example shows.

EXAMPLE 5.11. βN is an extremally disconnected space which is not

submaximal. Since $\beta N-N$ is closed in βN , it is quasi H -closed. However, $\beta N-N$ is not RS -compact.

COROLLARY 5.12. *Let X be a submaximal extremally disconnected space. Then, for a subspace A of X the following are equivalent:*

- (1) A is RS -compact.
- (2) A is S -closed.
- (3) A is nearly-compact.
- (4) A is quasi H -closed.

6. Functions and RS -compact spaces

It is known in [6] that RS -compactness is preserved under open and almost-continuous (in the sense of Singal [17]) surjections. In this section we obtain some improvements of the above result and investigate under what sort of functions the RS -compact property may be inverse-preserved. It will be pointed out that Theorem 2.4 of [6] is partially false, that is, although the RS -compact property is projective, it is not productive. We shall recall the definitions of some functions used in the subsequence.

A function $f: X \rightarrow Y$ is said to be *weakly-continuous* [7] (resp. θ -continuous [3], *almost-continuous* [17]) if for each point $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subset \text{Cl}(V)$ (resp. $f(\text{Cl}(U)) \subset \text{Cl}(V)$, $f(U) \subset \text{Int}(\text{Cl}(V))$). A function $f: X \rightarrow Y$ is said to be *almost-open* (resp. *weakly-open*) [15] if $f(U) \subset \text{Int}(\text{Cl}(f(U)))$ (resp. $f(U) \subset \text{Int}(f(\text{Cl}(U)))$) for every open set U of X .

REMARK 6.1. (1) Almost-continuity implies θ -continuity, θ -continuity implies weak-continuity and the converses are not necessarily true.

(2) In [15] the following are known: (a) Both almost-openness and weak-openness are implied by openness and they are independent of each other; (b) A function $f: X \rightarrow Y$ is almost-open if and only if for every open set V of Y $f^{-1}(\text{Cl}(V)) \subset \text{Cl}(f^{-1}(V))$.

THEOREM 6.2. *Let $f: X \rightarrow Y$ be an almost-open and weakly-continuous function. If K is RS -compact relative to X , then $f(K)$ is RS -compact relative to Y .*

Proof. Let $\{V_\alpha \mid \alpha \in \mathcal{V}\}$ be a cover of $f(K)$ by regular closed sets of Y . By Lemma 4.4 of [11], $\{f^{-1}(V_\alpha) \mid \alpha \in \mathcal{V}\}$ is a cover of K by regular closed sets of X . Since K is RS -compact relative to X , by Theorem 4.6 there exists a finite subset \mathcal{V}_0 of \mathcal{V} such that $K \subset \bigcup \{\text{Int}(f^{-1}(V_\alpha)) \mid \alpha \in \mathcal{V}_0\}$. Since f is almost-open, for each $\alpha \in \mathcal{V}_0$ we have

$$f(\text{Int}(f^{-1}(V_\alpha))) \subset \text{Int}(\text{Cl}(f(\text{Int}(f^{-1}(V_\alpha)))))) \subset \text{Int}(V_\alpha).$$

Therefore, we obtain that $f(K) \subset \bigcup \{\text{Int}(V_\alpha) \mid \alpha \in \mathcal{V}_0\}$. It follows from Theorem 4.6 that $f(K)$ is RS -compact relative to Y .

The following three corollaries are immediate consequences of Theorem 6.2. and the proofs are thus omitted.

COROLLARY 6.3. *RS -compactness is preserved under almost-open and weakly-continuous surjections.*

COROLLARY 6.4 (Hong [6]). *RS -compactness is preserved under open and almost-continuous surjections.*

COROLLARY 6.5 (Hong [6]). *If the product space is RS -compact, then each factor space is RS -compact.*

REMARK 6.6. In [6, Theorem 2.4], Hong stated that a nonempty product space is RS -compact if and only if each factor space is RS -compact. However, the sufficiency is false. By Remark 4.10, βN is RS -compact. Since $\beta N \times \beta N$ is not S -closed [9, p.193], by Theorem 2.3 $\beta N \times \beta N$ is not RS -compact. Therefore, the product of two RS -compact spaces is not necessarily RS -compact.

LEMMA 6.7 (Noiri [12]). *If a function $f : X \rightarrow Y$ is weakly-open and θ -continuous, then $f^{-1}(V)$ is regular open in X for every regular open set V of Y .*

THEOREM 6.8. *Let $f : X \rightarrow Y$ be a weakly-open and θ -continuous function. If K is RS -compact relative to X , then $f(K)$ is RS -compact relative to Y .*

Proof. By making use of Lemma 6.7 this is proved similarly to that of Theorem 6.2.

COROLLARY 6.9. *RS -compactness is preserved under weakly-open and*

θ -continuous surjections.

$RO(X)$ forms a base for a semi-regular topology. The set X with such a topology is called the *semi-regularization* of X and is denoted by X_s .

THEOREM 6.10. *A space X is RS -compact if and only if the semi-regularization X_s of X is RS -compact.*

Proof. Necessity. Let X be RS -compact and $i : X \rightarrow X_s$ be the identity function. For any open set U of X , $\text{Int}(\text{Cl}(U))$ is regular open in X and hence $i(\text{Int}(\text{Cl}(U)))$ is open in X_s . Therefore, we have

$$i(U) \subset i(\text{Int}(\text{Cl}(U))) \subset \text{Int}(i(\text{Cl}(U)))$$

for every open set U of X . Thus, i is weakly-open and continuous surjection. By Corollary 6.9, X_s is RS -compact.

Sufficiency. Let X_s be RS -compact. Since $i^{-1} : X_s \rightarrow X$ is an open and almost-continuous surjection, by Corollary 6.4 X is RS -compact.

REMARK 6.11. The images of RS -compact spaces under perfect mappings (continuous closed functions with compact point inverses) are not necessarily RS -compact.

EXAMPLE 6.12. By Remark 4.10, βN is RS -compact. By Example 2.4 of [9] $\bar{\tau} : \beta N \rightarrow \beta Q$ is perfect, not open and βQ is not S -closed. Therefore, by Theorem 2.3, βQ is not RS -compact.

A subset is called δ -open if the complement is δ -closed. A function $f : X \rightarrow Y$ is said to be δ -closed [10] if for each δ -closed set F of X $f(F)$ is δ -closed in Y .

LEMMA 6.13. *A surjection $f : X \rightarrow Y$ is δ -closed if and only if for every subset $S \subset Y$ and every δ -open set $U \subset X$ containing $f^{-1}(S)$, there exists a δ -open set $V \subset Y$ such that $S \subset V$ and $f^{-1}(V) \subset U$.*

Proof. The proof is parallel to that for closed functions and is thus omitted.

THEOREM 6.14. *Let $f : X \rightarrow Y$ be a δ -closed surjection and $f^{-1}(y)$ RS -compact relative to X for each $y \in Y$. If K is N -closed relative to Y , then $f^{-1}(K)$ is RS -compact relative to X .*

Proof. Let $\{V_\alpha | \alpha \in \mathcal{V}\}$ be a cover of $f^{-1}(K)$ by regular closed sets of X . For each $y \in K$, by Theorem 4.6 there exists a finite subset $\mathcal{V}(y)$ of \mathcal{V} such that $f^{-1}(y) \subset U(y)$, where $U(y) = \cup \{\text{Int}(V_\alpha) | \alpha \in \mathcal{V}(y)\}$. Since $U(y)$ is δ -open in X , by Lemma 6.13 there exists a δ -open set $V(y)$ of Y such that $y \in V(y)$ and $f^{-1}(V(y)) \subset U(y)$. The family $\{V(y) | y \in K\}$ is a cover of K by δ -open sets of Y . Since K is N -closed relative to Y , there exists a finite subset K_0 of K such that $K \subset \cup \{V(y) | y \in K_0\}$. Therefore, we obtain

$$f^{-1}(K) \subset \cup \{U(y) | y \in K_0\} = \cup \{\text{Int}(V_\alpha) | \alpha \in \mathcal{V}(y), y \in K_0\}.$$

It follows from Theorem 4.6 that $f^{-1}(K)$ is RS-compact relative to X .

COROLLARY 6.15. *Let $f : X \rightarrow Y$ be a δ -closed surjection and $f^{-1}(y)$ RS-compact relative to X for each $y \in Y$. If Y is nearly-compact, then X is RS-compact.*

Proof. This is an immediate consequence of Theorem 6.14.

COROLLARY 6.16. *Let $f : X \rightarrow Y$ be a δ -closed surjection and $f^{-1}(y)$ RS-compact relative to X for each $y \in Y$. If Y is RS-compact, then X is RS-compact.*

Proof. This follows from Theorem 2.3 and Corollary 6.15.

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