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A REPRESENTATION OF THE CHARACTERISTIC FUNCTIONAL OF SELF-DECOMPOSABLE PROBABILITY MEASURES ON CERTAIN *TV* SPACES

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1. Introduction

In paper [2], we defined a K-regular self-decomposable probability measure on a real locally convex Hausdorff topological vector (TV) space E and obtained several characterizations of these probability measures on E. These results subsume all similar known results obtained by Lévy [5] when E is the Euclidean space and by Kumar and Schreiber [4] when E is a real separable Banach space.

As a continuation of paper [2], this paper gives a characterization of K-regular self-decomposable probability measures on certain locally convex TV space in terms of their characteristic functionals. Our result here is motivated by the work of Kumar and Schreiber [4] where the Lévy-Khinchine representation of the characteristic functionals of self-decomposable probability measures defined on certain Orlicz sequence spaces is obtained. Crucial to the proof of our Theorem 3.1 are Lemmas 2.2 and 3.2 and the representation of the characteristic functionals of infinitely divisible probability measures on a complete Badrikian space obtained by Dettweiler [3].

2. Notation and Preliminaries

In this section we collect necessary notation, definitions and some known results which will be used in this paper.

Let E and E' denote a real Hausdorff locally convex TV space and its topological dual, respectively. By a probability (prob.) measure on E we will always mean that it is defined on $\mathscr{B}(E)$, the smallest σ-algebra containing the open sets of E. A prob. measure μ is K-

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regular if $\mu(B)=\sup_{K\subset B}\mu(K)$ for every $B\in\mathscr{B}(E)$, where K ranges over the compact subsets of B. $M_K(E)$ will denote the set of all K-regular prob. measures on E. If $\mu,\nu\in M_K(E)$, the convolution of μ and ν is defined by

$$\mu*\nu(B)=\int_E \mu(B-x)\nu(dx),$$

for every $B\in\mathscr{B}(E)$. For any $\mu\in M_K(E)$, and $a>0$, $T_a\mu$ is defined to be the prob. measure on E given by $T_a\mu(B)=\mu(a^{-1}B)$ for every $B\in\mathscr{B}(E)$.

Let $\langle\, ,\rangle$ be the natural bilinear form on $E\times E'$. For a prob. measure μ on E, the characteristic functional of μ, denoted by $\hat{\mu}(\cdot)$, is the function on E' defined by

$$\hat{\mu}(x')=\int_E e^{i\langle x,x'\rangle}\mu(dx)$$

for every $x'\in E'$. It is easily seen that, for every $\mu,\nu\in M_K(E)$, $\widehat{\mu*\nu}(\cdot)=\hat{\mu}(\cdot)\hat{\nu}(\cdot)$. It is well known [6] that every K-regular prob. measure on E is uniquely determined by its characteristic functional.

A subset $H\subset M_K(E)$ is called uniformly tight if, for every $\varepsilon>0$, there exists a compact subset K_ε of E such that $\mu(K_\varepsilon^c)<\varepsilon$, for all $\mu\in H$, where B^c denotes the complement of B. A measure $\mu\in M_K(E)$ is called *infinitely divisible* if, for each positive integer n, there exists a measure λ_n in $M_K(E)$ such that $\mu=\lambda_n^{*n}$.

DEFINITION 2.1. Let $\mu\in M_K(E)$. μ is called *self-decomposable* if for every $r\in(0,1)$, there exists a measure ν_r in $M_K(E)$ such that

$$\mu=T_r\mu*\nu_r.$$

The measure $\nu_r (0<r<1)$ will be called the component of μ.

LEMMA 2.2 ([2], p. 141). *Let $\mu\in M_K(E)$ be self-decomposable. Then μ and its components ν_r $(0<r<1)$ are infinitely divisible.*

Finally, we will note a few more notation, definitions and a theorem needed in the sequel (for details, see [3]). Let $\mathscr{K}$ denote the set of all compact subsets of E, and let $\mathscr{K}_0$ be the set of all compact convex circled subset of E. If $K\in\mathscr{K}_0$, then P_K will denote the Minkowski functional of K which is assumed to have $+\infty$ value off the space $E_K=\bigcup_{n=1}^{\infty} n\mathrm{K}$. Let $\mathscr{K}_H$ denote the set of all $\mathrm{K}\in\mathscr{K}_0$ such that P_K^2

is a positive quadratic form on the space E_K. If A is a subset of E, then A^0 denotes the polar of A relative to $\langle , \rangle$. The symbol $\sigma(E, E')$ will denote the weak topology on E generated by $\langle , \rangle$, and $\tau(E', E)$ will denote the Mackey topology (i. e., the topology of uniform convergence on all $\sigma(E, E')$-compact, convex circled subsets of E) on E'. E_c' will stand for the topological dual of E endowed with the topology of uniform convergence on the sets $K \in \mathcal{K}_0$.

DEFINITION 2. 3 ([3]). A locally convex TV space E is called a *Badrikian space* if $\mathcal{K}_H$ is a cofinal subset of $\mathcal{K}$; that is, for each $K \in \mathcal{K}$, there exists a $K_0 \in \mathcal{K}_H$ such that $K \subset K_0$.

It is known [3, p. 298] that every real Hilbert space and the dual of nuclear spaces are Badrikian spaces.

THEOREM 2. 4 ([3], p. 299). *Let E be a complete Badrikian space. If $\mu \in M_K(E)$ is infinitely divisible, then there exist*

(i) $x_0 \in E$,

(ii) *a continuous positive quadratic form Q on E_c',*

(iii) *a Lévy measure F, and*

(iv) $K \in \mathcal{K}_H$ *with* $\int_E \inf\ (P_K^2, 1)\quad F(dx) < \infty$ *such that, for every* $x \in E'$,

$$\hat{\mu}(x') = \exp\{i\langle x_0, x'\rangle - \frac{1}{2}Q(x') + \int_E \{e^{i\langle x, x'\rangle} - 1 - \frac{i\langle x, x'\rangle}{1 + P_K^2(x)}\} F(dx)\}. \tag{2.1}$$

Further, F and Q are uniquely determined by μ, and $x_0 \in E$ depends upon the choice of K.

Conversely, if a measure μ in $M_K(E)$ has the characteristic functional of the form (2. 1), *then μ is infinitely divisible.*

In view of the above theorem, for simplicity we will use the notation $\mu = [x_0, Q, F, \mathrm{K}]$ to denote the representation of an infinitely divisible measure μ in $M_K(E)$. We emphasize that if K is fixed in the representation, x_0, Q, and F are uniquely determined for a given infinitely divisible measure μ in $M_K(E)$.

3. Statement and the proof of the result

The purpose of this paper is to prove the following theorem:

THEOREM 3. 1. *Let E be a complete Badrikian space, and let $\mu \in M_K(E)$. Then, μ is self-decomposable if and only if*

$$\hat{\mu}(x') = \exp\{i\langle x_0, x'\rangle - \frac{1}{2}Q(x') + \int_E (e^{i\langle x, x'\rangle} - 1 - \frac{i\langle x, x'\rangle}{1+P_K^2(x)} F(dx)\}, \tag{3.1}$$

for every $x' \in E'$, where $x_0 \in E$, Q is a continuous positive quadratic form on E_c', F is a Lévy measure, and $K \in \mathcal{K}_H$ such that

$$\int_E \inf(P_K^2(x),\ 1) F(dx) < \infty, \tag{3.2}$$

and, for each $r \in (0,1)$, there exists a Borel measure F_r on E with

$$F = T_r F + F_r. \tag{3.3}$$

The proof of Theorem 3. 1 depends on the following Lemma.

LEMMA 3. 2. *Let h be the function of $E \times E'$ defined by*

$$h(x, x', K) = e^{i\langle x, x'\rangle} - 1 - \frac{i\langle x, x'\rangle}{1+P_K^2(x)},$$

where $K \in \mathcal{K}_H$. Then, for a fixed $r \in (0,1)$, there exists an $\bar{x} \in E$ such that, for every $x' \in E'$,

$$\int_E h(x, rx', K) F(dx) = \int_E h(x, x', K)\, T_r F(dx) + i\langle r(1-r^2)\bar{x}, x'\rangle, \tag{3.4}$$

where F is a Lévy measure on E, and $K \in \mathcal{K}_H$ is such that the condition (3. 2) is satisfied. (It may be noted that the left side of (3. 4) is finite).

Proof. We first observe that

$$\int_E h(x, rx', K) F(dx) = \int_E \{h(rx, x', K) + \frac{i\langle rx, x'\rangle}{1+P_K^2(rx)} - \frac{i\langle x, rx'\rangle}{1+P_K^2(x)}\} F(dx) \tag{3.5}$$

and recalling that $E_K = \bigcup_{n=1}^{\infty} nK$, we also have

$$\int_{E_K}\left\{\frac{\langle rx, x'\rangle}{1+P_K^2(rx)}-\frac{\langle x, rx'\rangle}{1+P_K^2(x)}\right\}F(dx) \tag{3.6}$$

$$=\int_{E_K}\left\{\frac{\langle rx, x'\rangle\cdot P_K^2(x)\cdot(1-r^2)}{(1+r^2P_K^2(x))(1+P_K^2(x))}\right\}F(dx)$$

$$\leq r(1-r^2)\left\{\int_{E_K}\frac{|\langle x, x'\rangle|P_K^2(x)}{(1+r^2P_K^2(x))(1+P_K^2(x))}F(dx)\right\}$$

$$\leq r(1-r^2)\left\{\int_K\frac{P_K^3(x)\cdot P_{K^0}(x')}{(1+r^2P_K^2(x))(1+P_K^2(x))}F(dx)\right\}$$

$$+\int_{E_K-K}\left\{\frac{P_K^3(x)\cdot P_{K^0}(x')}{(1+r^2P_K^2(x))(1+P_K^2(x))}\right\}F(dx)$$

$$(\text{note}\,|\langle x, x'\rangle|\leq P_K(x)\cdot P_{K^0}(x'))$$

$$\leq r(1-r^2)\left\{\int_K P_{K^0}(x')\cdot P_K^2(x)F(dx)\right.$$

$$\left.+\frac{1}{r^2}\int_{E_K-K}P_{K^0}(x')\,F(dx)\right\}$$

$$\leq r(1-r^2)P_{K^0}(x')\left\{\int_K P_K^2(x)F(dx)+\frac{1}{r^2}F(\{P_K>1\})\right\}$$

$$=r(1-r^2)P_{K^0}(x')\cdot c(r, K, F), \tag{3.7}$$

where

$$c(r, K, F)=\int_K P_K^2(x)F(dx)+\frac{1}{r^2}F(\{P_K>1\})<\infty;$$

$$(\text{as } (P_K>1)=K^c).$$

Now we shall show that, for every $x'\in E'$,

$$\int_E\left\{\frac{\langle rx, x'\rangle}{1+P_K^2(rx)}-\frac{\langle x, rx'\rangle}{1+P_K^2(x)}\right\}F(dx)<\infty.$$

To show that, since $P_K=\infty$ on $E_K{}^c$, it suffices to show that (3.6) is finite. From (3.7), this will follow if we can prove that $P_{K^0}(x')$ is finite for all $x'\in E'$. For every x' $(\neq 0)$ $\in E'$, let $t=\sup_{x\in K}|\langle x', x\rangle|$. Then, since K is compact, we see that $t>0$, and hence $|\langle x, \frac{1}{t}x'\rangle|\leq 1$, for all $x\in K$. This shows that x' belongs to tK^0, which implies that $P_{K^0}(x')<\infty$. Hence, by (3.5), we have

$$\int_E h(x, rx', K)F(dx)$$

$$=\int_E h(x, x', K)\,T_rF(dx)+ir(1-r^2).$$

$$=\int_E\frac{\langle x, x'\rangle P_K^2(x)}{(1+r^2P_K(x))(1+P_K^2(x))}F(dx). \tag{3.8}$$

Let E_τ' denote the topological dual of E endowed with Mackey topology $\tau(E', E)$. We define the functional ϕ on E_τ' by

$$x' \mapsto \phi(x') = \int_E \left\{ \frac{\langle x, x'\rangle P_K^2(x)}{(1+P_K^2(rx))(1+P_K^2(x))} \right\} F(dx).$$

Then, by the integrability of (3.6), ϕ is well-defined. Clearly, ϕ is linear. We now assert that ϕ is continous on E_τ'. To show this, let $\{x_\alpha'\}$ be any net in E_τ' with $x_\alpha' \to 0$ in $\tau(E', E)$-topology. Then, we need to show that $\phi(x_\alpha') \to \phi(\theta) = 0$. Since the topology $\sigma(E, E')$ on E is weaker than the original topology on E, it follows that K appearing in (3.4) is a $\sigma(E, E')$-compact. Hence, by the definition of the Mackey topology, we would have $\langle x, x_\alpha'\rangle \to \langle x, \theta\rangle = 0$ uniformly on K. Thus for any $\varepsilon > 0$, we can choose α_0 such that

$$|\langle x, x_\alpha'\rangle - \langle x, \theta\rangle| = |\langle x, x_\alpha'\rangle| < \varepsilon,$$

for all $\alpha \geq \alpha_0$, and, for all $x \in K$. This implies that $P_{K^0}(x') \leq \varepsilon$, for all $\alpha \geq \alpha_0$. Consequently, by (3.7)

$$|\phi(x_\alpha') - \phi(\theta)| = |\phi(x_\alpha')| \leq c(r, K, F) \cdot P_{K^0}(x') \leq c(r, K, F) \cdot \varepsilon,$$

for all $\alpha \geq \alpha_0$. Hence, $\phi \in (E_\tau')'$. However, since the Mackey topology is consistent with the duality $\langle\ ,\ \rangle$ [8, p. 131]; i. e., $(E_\tau')' = E$, we have an element $\bar{x} \in E$ such that $\langle \bar{x}, x'\rangle = \phi(x')$, for all $x' \in E_\tau'$. From (3.8), we easily see that $\bar{x}$ is the desired element in E which satisfies (3.4).

Proof of Theorem 3.1. Let μ be self-decomposable; then, for every $r \in (0, 1)$, there exists a $\nu_r \in M_K(E)$ such that

$$\mu = T_r\mu * \nu_r. \tag{3.9}$$

We know that in the proof of Theorem 3.7 of [2], $\{\nu_r\}_{r\in(0,1)}$ is a uniformly tight family of measures in $M_K(E)$. Since, by Lemma 2.2, μ and its components $\nu_r (0 < r < 1)$ are infinitely divisible, it follows that $\{\mu\} \cup \{\nu_r\}_{r\in(0,1)}$ is a uniformly tight family of infinitely divisible measures in $M_K(E)$. Hence, by Theorem 2.8 of [3, p. 301], we have a $K \in \mathscr{K}_H$ such that

$$\mu = [x_0, Q, F, K], \quad \int_E \inf\{P_K^2(x), 1\} F(dx) < \infty \tag{3.10}$$

and

$$\nu_r = [x_r, Q_r, F_r, K], \quad \int_E \inf\{P_K^2(x), 1\} F_r(dx) < \infty, \tag{3.11}$$

for all $r\in(0,1)$.

By using Lemma 3.2, we have, for each $r\in(0,1)$

$$\int_E\left\{e^{i\langle x,rx'\rangle}-1-\frac{i\langle x,rx'\rangle}{1+P_K^2(x)}\right\}F(dx)$$
$$=\int_E\left\{e^{i\langle x,x'\rangle}-1-\frac{i\langle x,x'\rangle}{1+P_K^2(x)}\right\}T_rF(dx)+i\langle r(1-r^2)\bar{x},x'\rangle \tag{3.12}$$

where $\bar{x}$ is the same element as in Lemma 3.2. Hence, from (3.10) and (3.12), we have

$$\widehat{T_r\mu}(x')=\hat{\mu}(rx')=\exp\left\{i\langle rx_0+r(1-r^2)\bar{x},x'\rangle-\frac{1}{2}r^2Q(x')\right.$$
$$\left.+\int_E\left\{e^{i\langle x,x'\rangle}-1-\frac{i\langle x,x'\rangle}{1+P_K^2(x)}\right\}T_rF(dx)\right\}.$$

Therefore, from (3.9), it follows that, for every $r\in(0,1)$,

$$[x_0,Q,F,K]=[r(x_0+(1-r^2)\bar{x},\ r^2Q,\ T_rF,K]\cdot[x_r,Q_r,F_r,K]$$
$$=[r(x_0+(1-r^2)\bar{x}+x_r,\ r^2Q+Q_r,\ T_rF+F_r,K].$$

By the uniqueness of the representation (note that K is the same for both representations), we must have $F=T_rF+F_r$, for every $r\in(0,1)$, which completes the proof of necessity.

To prove sufficiency, let $\hat{\mu}(\cdot)$ be of the form (3.1), and F satisfy (3.2) and (3.3). Then, by Theorem 2.4, μ is infinitely divisible. Further, since, from (3.3), $F_r\leq F$, for all $r\in(0,1)$, we can show that every F_r is a Lévy measure on E, and thus we have, for every $r\in(0,1)$,

$$\int_E\inf\{P_K^2(x),1\}F_r(dx)<\infty. \tag{3.13}$$

From (3.9), we easily see that $F_r(K^c)<\infty$, for all $r\in(0,1)$.

Now we note that, for each $r\in(0,1)$,

$$[x_0,Q,F,K]=[rx_0+r(1-r^2)\bar{x},r^2Q,T_rF,K]\cdot$$
$$[x_0-(rx_0+r(1-r^2)\bar{x}),\ (1-r^2)Q,F_r,K], \tag{3.14}$$

where $\bar{x}\in E$ is the same as before. Since F_r is a Lévy measure, and $F_r(K^c)<\infty$ for all $r\in(0,1)$, it can be shown [3] that there exists a unique infinitely divisible prob. measure ν_r $(0<r<1)$ in $M_K(E)$ such that $\nu_r=[x_0-rx_0-r(1-r^2)\bar{x},\ (1-r^2)Q,F_r,K]$. Hence, by (3.14), we conclude that

$$\mu = T_r \mu * \nu_r.$$

This completes the proof of the theorem.

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