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On Essential sub-$\wedge$-semimodules

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1. Introduction and preliminaries

M. Takahashi [3] introduced the notion of $\wedge$-semimodules and studied elementary properties f semimodules. In this paper, we study some properties of sub-$\wedge$-semimodules in section 2. ı section 3, we define essential sub-$\wedge$-semimodule and study its elementary properties.

A *semimodule* $A=(A,+,0)$ consists of a set A, a map $+: A\times A\longrightarrow A$ and an element 0 of . such that always

$$x+y=y+x,$$
$$(x+y)+z=x+(y+z),$$
$$x+0=x,$$

ɔr all $x,y,z\in A$.

A subset M of a semimodule A is a *sub-semimodule* of A if $0\in M$ and if $x,y\in M$ implies $+y\in M$.

A *semiring* $\wedge=(\wedge,+,0,\circ)$ consists of two data:

i) $(\wedge,+,0)$ is a semimodule,

ii) $(\wedge,\circ)$ is a semigroup,

uch that always

$$(\lambda+\mu)\circ\tau=\lambda\circ\tau+\mu\circ\tau,$$
$$\lambda\circ(\mu+\tau)=\lambda\circ\mu+\lambda\circ\tau,$$
$$0\circ\tau=\tau\circ 0=0.$$

Vhen there is no danger of confusion, we would denote $\lambda\circ\mu$ by $\lambda\mu$.

A semiring $\wedge$ is said to be *commutative* if $\lambda\mu=\mu\lambda$ for all $\lambda,\mu\in\wedge$.

A semiring $\wedge$ is said to *have an identity* if there exists $1\in\wedge$ such that $1\lambda=\lambda 1=\lambda$ or all $\lambda\in\wedge$.

Let $\wedge$ be a semiring with identity 1. *A left $\wedge$-semimodule* (or briefly *$\wedge$-semimodule*) A is a emimodule A together with a map $\eta:\wedge\times A\longrightarrow A$ written $\eta(\lambda,x)=\lambda x$, such that always

$$\lambda(x+y)=\lambda x+\lambda y,$$
$$(\lambda+\mu)x=\lambda x+\mu x,$$
$$(\lambda\mu)x=\lambda(\mu x),$$
$$1x=x,$$

$$\lambda 0=0x=0,$$

for all $\lambda, \mu\in\wedge$, $x, y\in A$.

For two $\wedge$-semimodules A and B, a $\wedge$-*homomorphism* $f: A\longrightarrow B$ is a map $f: A\longrightarrow B$ such that always

$$f(x+y)=f(x)+f(y),$$
$$f(\lambda x)=\lambda f(x),$$

for all $x, y\in A$, $\lambda\in\wedge$.

2. Sub-$\wedge$-semimodules

A subset L of a $\wedge$-semimodule A is a *sub-$\wedge$-semimodule* of A if $x, y\in L$ implies $x+y\in L$ and if $\lambda\in\wedge$, $x\in L$ implies $\lambda x\in L$.

A subset I of a semiring $\wedge$ is a *left ideal* in $\wedge$ if $I=(I, +, 0)$ is a sub-semimodule of $(\wedge, +, 0)$ and if $\wedge I\subseteq I$.

Proposition 2.1. *If L and N are sub-$\wedge$-semimodules of a $\wedge$-semimodule A, then their intersection $L\cap N$ is also a sub-$\wedge$-semimodule of A.*

Proof. If $x, y\in L\cap N$, then $x, y\in L$ and $x, y\in N$. Since L and N are sub-$\wedge$-semimodules of A, $x+y$, $\lambda x\in L$ and $x+y$, $\lambda x\in N$ for $\lambda\in\wedge$. Thus $x+y$, $\lambda x\in L\cap N$.

More generally, if $\{N_i | i\in I\}$ is an arbitrary nonempty family of sub-$\wedge$-semimodules of A, then $\bigcap_{i\in I} N_i$ is a sub-$\wedge$-semimodule of A.

Proposition 2.2. *Let $f: A\longrightarrow B$ be a homomorphism of $\wedge$-semimodules. If L is a sub-$\wedge$-semimodule of B, then $f^{-1}(L)=\{x\in A | f(x)\in L\}$ is a sub-$\wedge$-semimodule of A.*

Proof. If $x, y\in f^{-1}(L)$, then $f(x), f(y)\in L$. Then $f(x+y)=f(x)+f(y)\in L$ and $f(\lambda x)=\lambda f(x)\in L$ for $\lambda\in\wedge$. Hence $x+y\in f^{-1}(L)$ and $\lambda x\in f^{-1}(L)$.

Proposition 2.3. *Let I be a left ideal of a semiring $\wedge$ and let A be a $\wedge$-semimodule. Then, for $x\in A$,*

$$I_x=\{\lambda x | \lambda\in I\}$$

is a sub-$\wedge$-semimodule of A.

Proof. For any $\lambda x, \mu x\in I_x$, $\lambda x+\mu x=(\lambda+\mu)x\in I_x$ and also $\nu(\lambda x)=(\nu\lambda)x\in I_x$ *for all* $\nu\in\wedge$.

For two sub-$\wedge$-semimodules L and N of a $\wedge$-semimodule A, the sum of L and N, denoted by $L+N$, is defined by

$$L+N=\{x+y | x\in L \text{ and } y\in N\}$$

Then $L+N$ is obviously a sub-$\wedge$-semimodule of A.

Proposition 2.4. *Let L, M and N be sub-$\wedge$-semimodules of a $\wedge$-semimodule A. If $L\subseteq N$, then $L+(M\cap N)\subseteq(L+M)\cap N$.*

Proof. We have $L+(M\cap N)\subseteq L+M$. Also, since $L\subseteq N$, we obtain $L+(M\cap N)\subseteq N$. Hence $L+(M\cap N)\subseteq(L+M)\cap N$.

Proposition 2.5. *Let S be a subset of a commutative semiring $\wedge$ and let $\lambda\mu\in S$ whenever λ, $\mu\in S$. If N is a sub-$\wedge$-semimodule of a $\wedge$-semimodule A, then*

$$N_S=\{x\in A | \lambda x\in N \textit{ for some } \lambda\in S\}$$

is a sub-$\wedge$-semimodule of A.

Proof. If $x, y\in N_S$, then $\lambda x\in N$ and $\mu y\in N$ for some λ, $\mu\in S$. Then $\lambda\mu(x+y)=(\lambda\mu)x+(\lambda\mu)y=$

$\lambda) x+(\lambda\mu) y=\mu(\lambda x)+\lambda(\mu y) \in N$; also $\lambda(\nu x)=(\lambda\nu) x=(\nu\lambda) x=\nu(\lambda x) \in N$ *for* $\nu \in \Lambda$. Hence $x+y$, $\in N_S$ and we conclude that N_S is a sub-Λ-semimodule of A.

3. Essential sub-Λ-semimodules

A sub-Λ-semimodule M of a Λ-semimodule A is called *essential* of A if every nonzero sub Λ-semimodule of A has nonzero intersection with M.

Remark. A Λ-semimodule A is itself an essential of A.

Proposition 3.1. *If L and M are essential sub-Λ-semimodules of a Λ-semimodule A, then eir intersection $L \cap M$ is also essential of A.*

Proof. Let K be any nonzero sub-Λ-semimodule of A. Since M is an essential of A $M \cap K$ nonzero. Hence $L \cap M \cap K$ is nonzero since L is an essential. Therefore $L \cap M$ is an essential A.

Proposition 3.2. *Let L and M be two sub-Λ-semimodules of a Λ-semimodule A. If P and are essential sub-Λ-semimodules of L and M respectively, then $P \cap Q$ is an essential sub-Λ-mimodule of $L \cap M$.*

Proof. Clearly $P \cap Q$ is sub-Λ-semimodule of $L \cap M$. Let K be any nonzero sub-Λ-semimoule of $L \cap M$. Then $Q \cap K$ is nonzero since Q is an essential, whence $P \cap Q \cap K$ is nonzero since is an essential. Thus $P \cap Q$ is an essential sub-Λ-semimodule of $L \cap M$.

Proposition 3.3. *Let A be a Λ-semimodule, M a sub-Λ-semimodule of A and P a sub-Λ-emimodule of M. Then P is an essential of A if and only if P is an essential of M and M is n essential of A.*

Proof. First assume that P is an essential of M and M is an essential of A, Consider any onzero sub-Λ-semimodule K of A. Since M is an essential of A we have $K \cap M$ is nonzero, nd then since P is an essential of M we obtain $(K \cap M) \cap P$ is nonzero, that is, $K \cap P$ is nonero. Thus P is an essential of A.

Now suppose that P is an essential of A. Since any nonzero sub-Λ-semimodule of A has onzero intersection with P, the same can be said for nonzero sub-Λ-semimodules of M; hence P is an essential of M. Also, since any nonzero sub-Λ-semimodule K of A has nonzero intersction with P, $K \cap M$ is nonzero. Thus M is an essential of A.

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