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On Relational Spaces

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0. Introduction

Herrlich [3] has introduced the concept of topological categories and it is known that a topological category behaves almost like Set.

In this paper, we show that the category *Rel* of relational spaces and relation preserving maps a topological category, and characterize bireflective and subcategories of the *Rel*.

Moreover, we will give some internal characterizations of objects of those subcategories. Furermore it is shown that the category *RefRel* of reflexive relational spaces and relation preserving maps is a cartesian closed topological category. For the categorical terminology, we fer to [4, 5] and for the relational space, we refer to [1].

I. Relational spaces

In this section, we show that *Rel* is a topological category and *RefRel* is a cartesian closed pological category.

Definition 1.1. A *relational space* is a pair (X, R), where X is a set and R is a relation on .

Definition 1.2. Let (X, R) and (Y, S) be relational spaces and $f: X \longrightarrow Y$ a map. Then f is id to be a *relational preserving* map if for any $(x, y) \in R, f^2(x, y) \in S$, where f^2 denotes the map $\times f: X \times X \longrightarrow Y \times Y$.

Remark 1.3. (1) $f: (X, R) \longrightarrow (Y, S)$ is a relation preserving map if and only if $R \subseteq f^{-12}(s)$.
(2) Since one element set has the relation ϕ, every constant map f need not be a relation reserving map.

The following is immediate from definition 1.2.

Theorem 1.4. (1) *For any relational space* (X, R), *the identity map* $1_X: (X, R) \longrightarrow (X, R)$ *is relation preserving map.*
(2) *If* $f: \longrightarrow (X, R) \longrightarrow (Y, S)$ and $g: (Y, S) \longrightarrow (Z, T)$ *are relation preseving maps, then* $f: (X, R) \longrightarrow (Z, T)$ *is also a relation preserving map.*

It is clear by theorem 1.4 that the class of all relational spaces and relation preserving maps etween them forms a concrete category, which will be denoted by *Rel*.

Remark 1.5. Since every singleton set has two different relations, *Rel* is not properly fibred. But it is well-powered and co-(well-powered).

Theorem 1.6. *The category Rel is a topological and cotopological category. In particular,*

(1) *For a set X, a family* $((X_i, R_i))_{i\in I}$ *of relational spaces indexed by a class I, and a sourc* $(f_i : X \longrightarrow X_i)_{i\in I}$, *we define a relation R on X as follows:* $(x, y)\in R$ *if and only if for each* $i\in$ $f_i^2(x, y)\in R_i$. *Then R is the initial relation on X with respect to* $(f_i)_{i\in I}$. *In fact* $R=\bigcap_{i\in I} f_i^{-1^2}(R_i)$.

(2) *For a set X, a family* $((X_i, R_i))_{i\in I}$ *of relational spaces indexed by a class I and a sin* $(f_i : X_i \longrightarrow X)_{i\in I}$, *we define a relation R on X as follows*: $(x, y)\in R$ *if and only if for som* $i\in I$, *there is* $(h, k)\in R_i$ *with* $(x, y)=(f_i(h), f_i(k))$. *Then R is the final relation on X wit respect to* $(f_i)_{i\in I}$. *In fact* $R=\bigcup_{i\in I} f_i^2(R_i)$.

Proof. Let us show that *Rel* is a topological category. To do so, it is enough to show th the statement (1) holds. R is clearly a relation on X. By the definition of R, it is clear th for each $i\in I$, $f_i : (X, R)\longrightarrow(X_i, R_i)$ is a relation preserving map. Suppose for a relational spac (Y, S) and a map $g : Y\longrightarrow X$, $f_i g : (Y, S)\longrightarrow(X_i, R_i)$ is a relation preserving map for all $i\in$ Then for any $(x, y)\in S$, $f_i^2 g^2(x, y)=f_i^2(g(x), g(y))\in R$ for all $i\in I$. By the definition of R $g^2(s, y)\in R$ and hence $g : (Y, S)\longrightarrow(X, R)$ is a relation preserving map. It is known [5] th every topological category is cotopological. However, for the further development, we prove th second statement. It is obvious that R is a relation on X and for each $i\in I$, $f_i : (X_i, R_i)\longrightarrow$ (X, R) is a relation preserving map. Suppose for a relational space (Y, S) and a map $g : X\longrightarrow Y$ $gf_i : (X_i, R_i)\longrightarrow(Y, S)$ is a relational preserving map for all $i\in I$. Then for any $(x, y)\in R$ ther is an $i\in I$ and $(h, k)\in R_i$ with $f_i^2(h, k)=(x, y)$, since $g^2 f_i^2(h, k)=g^2(x, y)$ and gf_i is a relatio preserving map for all $i\in I$, $g(x, y)\in S$. Thus $g : (X, R)\longrightarrow(Y, S)$ is a relation preserving map This completes the proof.

The following is immediate from theorem in [5] and the above theorem.

Corollary 1.7. *Rel is complete and cocomplete.*

Definition 1.8. Let (X, R) be a relational space and A a subset of X. Then the initi relation R_A on A with respect to the inclusion map $A\longrightarrow X$ is called the *induced relation o A of R* and (A, R_A) is called the *subspace* of (X, R).

Definition 1.9. For a family $((X_i, R_i))_{i\in I}$ in *Rel* indexed by a set I, the initial relation R o $\prod X_i$ with respect to $(pr_i)_{i\in I}$ is called the *product relation* of $(R_i)_{i\in I}$, where pr_i denotes the it projection, and $(\prod X_i, R)$ is called the *product space of the family.*

Remark 1.10. Let $((X_i, R_i))_{i\in I}$ be a family in *Rel* and R the product relation of $(R_i)_{i\in I}$. The $((x_i), ((y_i))\in R$ if and only if for each $i\in I$, $(x_i, y_i)\in R_i$. In fact $R=\prod R_i$.

In the following, the set of all relation preserving maps on a relational space $\underline{X}$ to a rel tional space $\underline{Y}$ will be denoted by $\hom(\underline{X}, \underline{Y})$.

Notation. *RefRel* denotes the full subcategory of *Rel* determined by those objects with refle xive relations.

Remark 1.11. For any $(X, Y)\in Rel$, the identity map $1_X : (X, R)\longrightarrow(X, R\cup\Delta_X)$ gives rise t the *RefRel*-reflection of (X, R). Hence *RefRel* is bireflective in *Rel* and hence topological (see[5])

Theorem 1.12. *RefRel is a cartesian closed category.*

Proof. Since *RefRel* is a topological category, it has product. Take $\underline{X}=(X, R_X)\in Rel$. For ny $\underline{Y}=(R_Y)\in Rel$, we define a relation R on $\underline{Y}^{\underline{X}}=hom(\underline{X}, \underline{Y})$ as follows: $(f, g)\in R$ if and only f for any $(a, b)\in R_X$, $(f(a), g(b))\in R_Y$. Then it is obvious that $(\underline{Y}^X, R)\in Rel$. Now we define $_{X,Y}: \underline{X}\times \underline{Y}^Y \longrightarrow \underline{Y}$ by $e_{X,Y}(a, f)=f(a)$ $(a\in X, f\in \underline{Y}^X)$. Let S be the relation on the product space $\underline{X}\times \underline{Y}^X$. Suppose $((a, f), (b, g))\in S$. Since $(a, b)\in R_X$ and $(f, g)\in R$, $e_{X,Y}((a, f), (b, g))=(f(a),$ $g(b))\in R_Y$. Hence $e_{X,Y}$ is a relation preserving map on $\underline{X}\times \underline{Y}^X$. Now any $g: \underline{X}\times \underline{Z}\longrightarrow \underline{Y}$ in *Rel*, ve define $\bar{g}: \underline{Z}\longrightarrow \underline{Y}^X$ *by* $\bar{g}(c)(a)=g(a, c)$ $(a\in X, c\in Z)$. Since $Z\in RefRel$, $\bar{g}$ is well-defined. Let R_Z be the relation on Z and $R_X\times R_Z$ the product relation on $X\times Z$. Suppose $(c, c')\in R_Z$ and $(a, b)\in R_X$. Then since g is a relation preserving map on $X\times Z$, $(\bar{g}(c)(a), \bar{g}(c')(b))=(g(a, c),$ $g(b, c'))\in R_Y$. Hence by the definition of R, $(\bar{g}(c), \bar{g}(c'))\in R$. Thus $\bar{g}$ is a relation preserving nap on Z. Then for any $(a, c)\in X\times Z$, $e_{X,Y}(1_X\times \bar{g})(a, c)=e_{X,Y}(a, \bar{g}(c))=g(c)(a)=g(a, c)$. Hence $e_{X,Y}(1_X\times \bar{g})g$. Oviously such a $\bar{g}$ is unique. By theorem in [2], this completes the proof.

II. Bireflective and epireflective subcategories of *Rel*

In this section, we characterize bireflective and epireflective subcategories of *Rel* and we will give some internal characterization of objects of those subcategories.

Notation. (1) *Equiv* denotes the full subcategory of *Rel* determined by all objects (X, R), where R is an equivalence relation on X.

(2) *Qord* denotes the full subcategory of *Rel* determined by all objects (X, R), where R is a quasi order relation on X.

(3) *Poset* denotes the full subcategory of *Rel* determined by all objects (X, R), where R is a partial order relation on X.

The following theorem follows from theorem in [3].

Theorem. 2.1. *Let B a full, isomorphism closed subcategory of a topological category A. Then*

(1) *B is bireflective in A if and only if B is closed under initial sources.*

(2) *B is epireflective in A if and only if B is closed under inititial momo-sources.*

Theorem 2.2. *Equiv and Qord are bireflective in Rel.*

Proof. It is enough to show that both categories are closed under initial sources. Suppose $(f_i: (X, R)\longrightarrow (X_i, R_i))_{i\in I}$ is an initial source in *Rel* such that for all $i\in I$, (X_i, R_i) belongs to *Equiv* (*Qord*, resp.). Since for each $i\in I$, R_i is an eqivalence(quasi order, resp.) relation on X_i and $f_i^2(x, x)\in R_i (x, x)\in R$, i. e. R is reflexive. Suppose $(x, y)\in R$. Then $f_i^2(x, y)\in R_i$ for all $i\in I$. Since for each $i\in I, R_i$ is symmetric, $f_i^2(y, x)\in R_i$ for all $i\in I$. By the definition of R, $(y, x)\in R$. Hence R is aymmetric. Suppose $(x, y)\in R$ and $(y, z)\in R$. Then for each $i\in I$, $f_i^2(x, y)$ $\in R_i$ and $f_i^2(y, z)\in R_i$. Since for each $i\in I, R_i$ is tranistive, $f_i^2(x, z)\in R_i$ for all $i\in I$, so that $(x, z)\in R$. Hence R is tranistive. This completes the proof.

Remark 2.3. For any $(X, R)\in Rel$, the identity map $1_X: (X, R)\longrightarrow (X, R')$ gives rise to the *Equiv* (*Qord*, resp.)-reflection of (X, R), where R' is the smallest equivalence (quasi order, sesp.) relation on X with $R\subseteq R'$

Corollary 2.4. *Equiv and Qord are properly fibred topological categories.*

Proof. It follows from the fact that each singleton set has unique reflexive relation.

Theorem 2.5. *Equiv and Qord are cartesian closed categories.*

Proof. Since *Equiv* and *Qord* and topological categories. They have products. Take $\underline{X}=(X, R_X)\subset Equiv\,(Qord,$ resp.). For any $\underline{Y}=(Y, R_Y)\in Equiv\,(Qord,$ resp.), let R be the relation on $\underline{Y}^X=\hom(\underline{X}, \underline{Y})$. Then it is straightforward that $(\underline{Y}^X, R)$ belongs to *Equiv* (*Qord*, resp.). Hence, by the same argument as that in theorem 1.12, *Equiv* and *Qord* are both cartesian closed.

Remark. 2.6. In the category *Equiv* (*Qord*, resp.), let us define a relation R' on $\hom(X, Y)$ as follows: $(f, g)\in R'$ if and only if for any $x\in X$, $(f(x), g(x))\in R_Y$, where R_Y is the relation on Y. Then R$'$ is equal to the relation R in the proof of theorem 2.5.

Theorem 2.7. *Poset is epireflective in Rel.*

Proof. It is enough to show that *Poset* is closed under initial momo-sources. Suppose $(f_i : X \longrightarrow (X_i, R_i))_{i\in I}$ is an initial momo-source in *Rel* such that for all $i\in I$, (X_i, R_i) belongs to *Poset*. Let R be the initial relation on X with respect to $(f_i)_{i\in I}$. Then we have to show that R is a partial order relation on X. Then by the exactly same as that in the proof of theorem 2.2, R is reflexive and transitive. It remains to show that R is anti-symmetric. Suppose $(x, y)\in R$ and $(y, x)\in R$. Then for each $i\in I$, $f_i^2(x, y)\in R_i$ and $f_i^2(y, x)\in R_i$. Since for each $i\in I$, R_i is anti-symmetric, $f_i(x)=f_i(y)$ for all $i\in I$. Since $(f_i)_{i\in I}$ is a momo-source, one has $x=y$.

Theorem. 2.8. *Let (X, R) be a relation space. Then the following are equivalent:*

(1) *(X, R) belongs to Equiv.*

(2) *$hom((X, R), \underline{T})$ is an initial momo-source, where $T=[\{0, 1, 2\}, \Delta\cup\{(1, 1), (1, 1)\}]$.*

(3) *(X, R) is isomorphic with a subspace of a power of T.*

Proof. (1)$\Longrightarrow$(2). Let R' be the initial relation on X with respect to $\hom(\underline{X}, \underline{T})$. Then it is obvious that $R\subseteq R'$. Suppose $(x, y)\in R$ and let $f_x : \underline{X}\longrightarrow\underline{T}$ be the map defined by

$$f_x(z)=\begin{cases}1 & \text{if } (x, z)\in R\\ 2 & \text{if } (x, z)\notin R.\end{cases}$$

Then it is obvious that f_x is a relation preserving map on X. Since $(x, y)\notin R$, $f_x^2(x, y)\notin R_T$, where $R_T=\{\Delta\cup\{(0, 1), (1, 0)\}\}$. Hence $(x, y)\notin R'$ and hence $R'\subseteq R$; therefore $R=R'$. Now it remains to show that $\hom(\underline{X}, \underline{Y})$ is a momo-source. Suppose $x\neq y$ in X and $f : \underline{X}\longrightarrow\underline{T}$ be the map defined by $f(x)=1$ and $f(X-\{x\})=0$. Then it is obvous that f is a relation preserving map on X and $f(x)\neq f(y)$.

(2)$\Longrightarrow$(3). Let $h : \underline{X}\longrightarrow\underline{T}^{\hom(X, T)}$ be the map defined by $h(x)=(f(x))_{f\in\hom(X, T)}$. Since $\hom(\underline{X}, \underline{T})$ is an initial momo-source, h is an embedding.

(3)$\Longrightarrow$(1). Since *Equiv* is closed under initial momo-source, amd $\underline{T}\in Equiv$, one has $(X, R)\in$ *Equiv*.

Using the exactly same arqument as that in theorem 2.6, we have the following.

Theorem 2.9. *Let (X, R) be a relation space. Then the following are equivalent:*

(1) *(X, R) belongs to Qord.*

(2) *$hom((X, R), \underline{V})$ is an initial momo-source, where $V=\{\{0, 1, 2\}, V\cup\{(0, 1), (1, 0), (2, 0), (2, 1)\}\}$.*

(3) *(X, R) is isomorphic with a subspace of a power of* $\underline{V}$.

Theorem. 2.10. *Let (X, R) be a relational space. Then the following are equivalent:*

(1) *(X, R) belongs to Poset.*

(2) *$hom((X, R), \underline{I})$ is an initial momo-source, where* $\underline{I}=\{\{(0,1)\}, \Delta\cup\{(0,1)\}\}$.

(3) *(X, R) is isomorphic with a subspace of a power of* $\underline{I}$.

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