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A Study on Interpolating Sequences

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1. Introduction

Let H^∞ be the Banach algebra of bounded analytic functions on the unit disc D in the complex plane. We denote by $M(H^\infty)$ the set of complex homomorphisms of H^∞. $M(H^\infty)$ is called the maximal ideal space of H^∞. It is well known [5, p. 268] that $M(H^\infty)$ is a compact Hausdorff space with respect to the weak star topology, and $\|m\|=m(1)=1$ for each complex homomorphism m in $M(H^\infty)$.

Writing

$$\hat{f}(m)=m(f),\ f\in H^\infty, m\in M(H^\infty),$$

we have a homomorphism $f\to\hat{f}$ from H^∞ into $C(M(H^\infty))$, the algebra of continuous complex-valued functions on $M(H^\infty)$. This homomorphism is called the Gelfand transform. Let $\triangle$ be the set of point evaluations, that is, the complex homomorphisms m_λ in $M(H^\infty)$ defined by $m_\lambda(f)=f(\lambda)$, and let Z be the indentity function on D. Then it follows a theorem [4, p. 160] that $\hat{Z}$ maps $\triangle$ homeomorphically onto D.

A function of the form

$$B(Z)=Z^k\prod_{n=1}^{\infty}\frac{|Z_n|}{Z_n}\frac{Z_n-Z}{1-\bar{Z}_nZ}$$

is called a Blaschke product. Here k is a nonnegative integer and $\{Z_n\}$ is a sequence in D such that $Z_n\neq0$ and $\sum(1-|Z_n|)<\infty$. $B(Z)$ converges uniformly on compact subsets of D and is bounded in modulus by 1 and so is an element of H^∞ with sup norm 1 [6, p. 333].

A sequence $\{Z_n\}$ in D is called an interpolating sequence if, for every bounded sequence of complex numbers $\{W_n\}$, there exists a function f in H^∞ such that $f(Z_n)=W_n$ for every n.

A geometric characterization of interpolating sequences is expressed in terms of the pseudo-hyperbolic metric ρ, defined by

$$\rho(a,b)=\left|\frac{a-b}{1-\bar{a}b}\right|,\ a,b\in D.$$

Carleson's interpolation theorem [1] states that a sequence $\{Z_n\}$ in D is an interpolating sequence if and only if

$$\inf_n\prod_{k\neq n}\rho(Z_n,Z_k)>0$$

In this paper, we consider the hull of sequences in $M(H^\infty)$, and investigate the relationship between the maximal ideal space of H^∞ and interpolating sequences.

2. Main results

Let $S=\{m_k\}$ be a sequence of distinct points in $M(H^\infty)$. By the hull of S, we mean the set of all points m in $M(H^\infty)$ such that $\hat{f}(m)=0$ for every $\hat{f}$ which vanishes on S.

Proposition 1. *If $S=\{Z_k\}$ is a sequence of distinct points in D, then the following statements are equivalent.*

(*a*) *hull* $(S)=\bar{S}$.

(*b*) *If B is a Blaschke product with zeros $\{Z_k\}$, then every zero of $\hat{B}$ on $M(H^\infty)$ is in $\bar{S}$.*

Proof. First, suppose that (a) holds, and that $f=0$ on S. Put $f=Bg$. If $\hat{B}(m)=0$, then $\hat{f}(m)=\hat{B}(m)\hat{g}(m)=0$. Hence m belongs to *hull* $(S)=\bar{S}$.

We now show that (b) implies (a). It is clear that $\bar{S}\subset$ *hull*(S). Suppose $m\in$ *hull*(S). Since $B=0$ on S, it follows that $\hat{B}(m)=0$; hence, $m\in\bar{S}$, by (b). Consequently, we have *hull*$(S)=\bar{S}$.

Let X be a compact Hausdorff space. For each x in X we define the complex function m_x on $C(X)$ by $m_x(f)=f(x)$ for every f in $C(X)$. It is immediate that m_x is in $M(C(X))$, the maximal ideal space of the Banach algebra $C(X)$.

Lemma 2. *Let φ denote the mapping from X to $M(C(X))$ defined by $\varphi(x)=m_x$. Then φ defines a homeomorphism from X onto $M(C(X))$*, where $M(C(X))$ is given the relative weak star topology on $C(X)^*$.

Proof. See [2, p. 33].

Let l^∞ denote the space of bounded complex sequences. With the norm $\|x\|=\sup_n|x_n|$ and with the pointwise multiplication $(xy)_n=x_ny_n$, l^∞ is a Banach algebra. The maximal ideal space of l^∞ has the special name βN, the Stone-Čech compactification of the positive integers N. The Stone-Cech compactification βN can be

Lemma 3. *Let X be a compact Hausdorff space and let $\tilde{\tau}: N\longrightarrow X$ be a continuous mapping. Then the mapping τ has a unique continuous extension $\tau:\beta N\longrightarrow x$. If $\tau(N)$ is dense in X and if the images of disjoint subsets of N have disjoint closures in X, then the extension $\tilde{\tau}$ is a homeomorphism of βN onto X.*

Proof. See [3, p. 186].

Theorem 4. *If $S=\{Z_k\}$ is a sequence of distinct points in the open unit disc D, then the following statements are equivalent.*

(*a*) *S is an interpolating sequence.*

(*b*) *Every idempotent sequence can be interpolated.*

(*c*) *$\bar{S}$ in $M(H^\infty)$ is homeomorphic to the Stone-Čech compactification of the positive integers, and if B is the Blaschke product with zeros $\{Z_k\}$, then every zero of $\hat{B}$ on $M(H^\infty)$ is in $\bar{S}$.*

Proof. First, we shall suppose that (a) holds and let $W=\{W_k\}$ be an idempotent sequence. Then $H^\infty|S=l^\infty$ and $W\in l^\infty$. Hence, there exists a function f in H^∞ such that $f(Z_k)=W_k$ for all k. This proves that condition (a) implies (b).

We shall show that (b) implies (a). We note that $\hat{H}^\infty|S$ is a complex linear subalgebra of l^∞ which contains every idempotent in l^∞. Let I be the clesed ideal consisting of those element f in H^∞ such that $\hat{f}=0$ on S. Then $f+I\longleftrightarrow\hat{f}|S$ is an isomorphism between the quotient algebra

H^∞/I and the algebra $\hat{H}^\infty|S$. The standard quotient norm on H^∞/I is a Banach algebra norm; hence $\|\hat{f}|S\| = \|f+I\|$ is a Banach algebra norm on $\hat{H}^\infty|S$. Consequently, we have $\hat{H}^\infty|S=l^\infty$. Hence (b) implies (a).

We now prove that (a) implies (c). Since $H^\infty|S=l^\infty$, it follows that every bounded function on the sequence S is the restriction to S of a continuous function $\hat{f}$; hence each Z_k is isolated from the points Z_j, $j \neq k$. In other words, S is discrete as a topological subspace of $M(H^\infty)$. Therefore, S is homeomorphic to the positive integers. Each bounded function on S has a continuous extension to $\bar{S}$, by (a). The functions $\hat{f}, f$ in H^∞, separate the points of $\bar{S}$, since they separate the points of $M(H^\infty)$. Since S is dense in $\bar{S}$, it follows from Lemma 3 that $\bar{S}$ is homeomorphic to the Stone-Čech compactification of the positive integers. For any m in the hull of S, the mapping $\sigma : \hat{f}|S \to f(m)$ defines a complex homomorphism of the algebra $\hat{H}^\infty|S$. But $\hat{H}^\infty|S=l^\infty$, which is isomorphic to the algebra $C(\bar{S})$ of all continuous functions on $\bar{S}$. Therefore, σ belongs to the algebra $C(\bar{S})$. Hence, by Lemma 2, σ is evaluation at a point of $\bar{S}$, that is, m is in $\bar{S}$. We conclude that $\bar{S}=hull(S)$. So, by Proposition 1, every zero of $\hat{B}$ on $M(H^\infty)$ is in $\bar{S}$.

To prove that (c) implies (b), let I be the closed ideal consisting those elements f in H^∞ such that $\hat{f}=0$ on S. Then $f+I \longleftrightarrow \hat{f}|S$ is an isomorphism between H^∞/I and $H^\infty|S$. $\hat{H}$ence $\hat{H}^\infty|S$ is a commutative Banach algebra, using the norm inherited from $\hat{H}^\infty/I$. It is immediate that every complex homomorphism of $\hat{H}^\infty|S$ is evaluation at a point of the hull of S. Since hull $(S)=\bar{S}$ and $\bar{S}$ is the stone-Čech compactification of the discrete countable space S, the maximal ideal space of $\hat{H}^\infty|S$ is the totally disconnected space $\bar{S}$. By the theorem of Shilov, $\hat{H}^\infty|S$ contains every idempotent continuous function on $\bar{S}$. Hence, we have (b).

REFERENCES

1. L. Carleson, An interpolation theorem for bounded analytic functions, *Amer. J. Math.* 80 (1958), 921~930.
2. R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.
3. J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.
4. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, New Jersey, 1962.
5. W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.
6. W. Rudin, *Real and Complex Analysis*, 2nd ed., McGraw-Hill, New York, 1974.