韓國數學教育學會誌<數學教育>
985. 12, Vol. XXIV, No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1985, Vol. XXIV, No. 1

On the Inverse Transform of the Positive part of a Fourier Transform

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1. Introduction

Let $\mathcal{F}$ denote the Fourier transformation; for $L^1(\mathbf{R}^n)$, $\hat{f}(\xi)=\mathcal{F}f(\xi)=\int_{R^n} f(x)e^{-2\pi ix\xi}dx, \xi\in\mathbf{R}^n$.
t is well known that f can be obtained from $\hat{f}$ by the inverse transformation $\mathcal{F}^{-1}$ under some ıypothesis on $\hat{f}$, here $(\mathcal{F}^{-1}g)(x)=\mathcal{F}g(-x)$.

Now we decompoes $\hat{f}$ so that

$$\hat{f}(\xi)=[r^+(\xi)-r^-(\xi)]+i[s^+(\xi)-s^-(\xi)]$$

where $r^+(\xi)=max(Re(\hat{f}), 0)$.

Here we consider $\mathcal{F}^{-1}(r^+)$ and ask whether $\mathcal{F}^{-1}(r^+)\in L^1(\mathbf{R}^n)$ when f satisfies some regularity onditions. The answer depends on the space dimension n: if $n\leqq 2$ then mild conditions on f ssures that $\mathcal{F}^{-1}(r^+)\in L^1(\mathbf{R}^n)$, while if $n\geqq 3$ then $\mathcal{F}^{-1}(r^+)\notin L^1(\mathbf{R}^n)$ in general. In what follows ve maintain the notation r^+ for the positive part of $\hat{f}$ and r denotes the real part of $\hat{f}$.

2. Main results

Proposition 1. *If* $x\longrightarrow(1+|x|)f(x)$ *is in* $L^2(\mathbf{R}^1)$ *then* $\mathcal{F}^{-1}(r^+)\in L^1(\mathbf{R}^1)$ *and*

$$||F^{-1}(r^+)||_{L^1}\leqq\frac{1}{\sqrt{2}}\left[\int_R(1+4\pi^2x^2)|f(x)|^2dx\right]^{\frac{1}{2}}.$$

Proof. From the hypothesis we know that
$f\in L^1\cap L^2$, so that $\hat{f}\in C_o$ and $r=Re(\hat{f})\in C_o$ and $\hat{f}\in L^2$ and $r^+\in L^2$. Let $D=\{\xi\in\mathbf{R}|r(\xi)>0\}$. 'hen $r^+=\chi_D r$ where χ_D is a characteristic function on D. In distribution sense,

$$(r^+)'=(\chi_D)'r+\chi_D r' \text{ and}$$

$r=0$ on which $\chi_D'\neq 0$. Hence

$$(r^+)'=\chi_D r'.$$

Jsing this equality and Cauchy-Schwarz inequality and Plancherel theorem,

$$\int_R|\mathcal{F}^{-1}(r^+)|dx\leqq\left[\int_R(1+4\pi^2x^2)^{-1}dx\right]^{\frac{1}{2}}\left[\int_R(1+4\pi^2x^2)|\mathcal{F}^{-1}r^+|^2dx\right]^{\frac{1}{2}}$$

$$=\frac{1}{\sqrt{2}}\left[\int_R(1+4\pi^2x^2)|\mathcal{F}^{-1}r^+|^2dx\right]^{\frac{1}{2}}$$

n the other hand,

$$\int_R (|\mathcal{F}^{-1}r^+|^2+4\pi^2x^2|\mathcal{F}^{-1}r^+|^2)\,dx$$
$$=\int_R (|\mathcal{F}^{-1}r^+|^2+|2\pi ix\mathcal{F}^{-1}r^+|^2)\,dx$$
$$=\int_R (|\mathcal{F}^{-1}r^+|^2+|\mathcal{F}^{-1}(r^+)'|^2)\,dx$$
$$=\int_R (|r^+|^2+|(r^+)'|^2)\,d\xi \leqq \int_R (|r(\xi)|^2+|r'(\xi)|^2)\,d\xi$$
$$\leqq \int_R (|\hat{f}(\xi)|^2+|\hat{f}'(\xi)|^2)\,d\xi$$
$$=\int_R (|f(x)|^2+|-2\pi ixf(x)|^2)\,dx$$
$$=\int_R (1+4\pi^2x^2)|f(x)|^2dx$$

combining these two inequalities, we obtain the required result. (Q. E. D.)

Before considering the two dimensional case, we need some lemma, due to M, Cowling.

Lemma 2. *Suppose that* $r: R \longrightarrow R$ *is in* $C^2(R)$ *and that* r *and* $r' \in C_o(R)$. *If* $u(x)=\int_R r^+(\xi)e^{2\pi ix\xi}d\xi$, *then*

$$|u(x)| \leqq \frac{3}{8\pi^2x^2}\int_R |r''(\xi)|\,d\xi$$

Proof. Let $D=\{\xi \in R \mid r(\xi)>0\}$, which is open in R. Then $u(x)=\int_D r(\xi)e^{2\pi ix}d\xi$.

We may write; $D=\bigcup_{n=1}^{\infty} I_n$, $I_n=(a_n, b_n)$, I_n's are disjoint.

We assume that I_n's are finite intervals for a moment.

Then $u(x)=\sum_{n=1}^{\infty}\int_{I_n} r(\xi)e^{2\pi ix\xi}d\xi$.

Integration by parts, together with the fact that

$$r(a_n)=0=r(b_n),$$

shows that

$$u(x)=-\sum_n^{\infty}(2\pi ix)^{-1}\int_{I_n} r'(\xi)e^{2\pi ix\xi}d\xi$$
$$=-\sum_n^{\infty}(2\pi ix)^{-2}\left[r'(\xi)e^{2\pi ix\xi}\Big|_{a_n}^{b_n}-\int_{I_n} r''(\xi)e^{2\pi ix\xi}d\xi\right].$$

So

$$|u(x)| \leqq \sum_n (2\pi x)^{-2}\left[|r'(a_n)|+|r'(b_n)|+\int_{I_n}|r''(\xi)|\,d\xi\right]$$

Now it is enough to show that

$$|r'(a_n)|+|r'(a_n)| \leqq \frac{1}{2}\int_{I_n}|r''(\xi)|\,d\xi.$$

Since $r(a_n)=0$ and $r(a_n+\varepsilon)>0$ for small $\varepsilon>0$, $r'(a_n) \geqq 0$. Similarly $r'(b_n) \leqq 0$.

If $r'(a_n)=0$, then put $a_n'=a_n$ and otherwise, put $a_n'=sup\{\xi \in R \mid \xi<a_n,\ r'(\xi)=0\}$.

Similarly if $r'(b_n)=0$ then, put $b_n'=b_n$ and otherwise, put $b_n'=inf\{\xi \in R \mid \xi>b_n,\ r'(\xi)=0\}$.

Furthermore, there exists a number $c_n \in (a_n, b_n)$ such that $r'(c_n)=0$ by Rolle's theorem. Let $I_n'=(a_n', b_n')$. By construction I_n's are disjoint intervals and so

$$r'(a_n)=\int_{a_n'}^{a_n} r''(\xi)\,d\xi=-\int_{a_n}^{c_n} r''(\xi)\,d\xi,$$

$$2|r'(a_n)|\leqq\int_{a_n'}^{a_n}|r''(\xi)|d\xi+\int_{a_n}^{c_n}|r''(\xi)|d\xi.$$

ilarly,

$$2|r'(b_n)|\leqq\int_{c_n}^{b_n}|r''(\xi)|d\xi+\int_{b_n}^{b_n'}|r''(\xi)|d\xi.$$

m these two inequalities,

$$|u(x)|\leqq(2\pi x)^{-2}\sum_n\left[\frac{1}{2}\int_{I_n'}|r''(\xi)|d\xi+\int_{I_n}|r''(\xi)|d\xi\right]$$

$$\leqq(2\pi x)^{-2}\left[\frac{1}{2}\int_R|r''(\xi)|d\xi+\int_R|r''(\xi)|d\xi\right]$$

$$=\frac{3}{8\pi^2x^2}\int_R|r''(\xi)|d\xi,$$

ch is the required result.

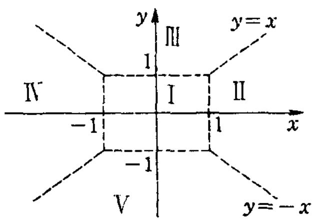
ut if some I_n is infinite, the above arguments no longer make sense, thus a slight moditon is needed, but; since r and r' are in $C_o(R)$ the technique used above is valid for each ɔ. (Q. E. D.)

Iow we consider the case when $n=2$.

'roposition 3. *Suppose that the functions* $(x, y)\longrightarrow(1+x^2+y^2)f(x, y)$, $\frac{\partial}{\partial x}\left(x^2f(x, y)\right)$ *and* $\frac{\partial}{\partial y}\left(y^2f(x,y)\right)$ *are in* $L^2(R^2)$.
en $\mathcal{F}^{-1}(r^+)\in L^1(R^2)$ *and*

$$\|\mathcal{F}^{-1}(r^+)\|_{L^1}\leqq2\|f\|_{L^2}+6\|x^2f+\frac{\partial}{\partial x}(x^2f)\|_{L^2}+6\|y^2f+\frac{\partial}{\partial y}(y^2f)\|_{L^2}.$$

Proof. We may devide R^2 into 5 regions;



the region I, Cauchy-Schwarz inequality gives

$$\int_{-1}^1\int_{-1}^1|\mathcal{F}^{-1}(r^+)(x, y)|dxdy\leqq2\left[\int_{-1}^1\int_{-1}^1|\mathcal{F}^{-1}(r^+)|^2dxdy\right]^{\frac{1}{2}}$$

$$\leqq2\|r^+\|_{L^2}\leqq2\|\hat{f}\|_{L^2}=2\|f\|_{L^2}.$$

ɔw we shall show that in the region II

$$\int_1^\infty\int_{-x}^x|\mathcal{F}^{-1}(r^+)(x,y)|dxdy\leqq3\|x^2f+\frac{\partial}{\partial x}(x^2f)\|_{L^2}.$$

ｒ symmetry analogous estimations hold for the regions II ∼ V, whence the theorem follows. ɔserve that

$$\int_1^\infty\int_{-x}^x|\mathcal{F}^{-1}(r^+)(x,y)|dxdy$$

$$\leqq\int_1^\infty (2x)^{\frac{1}{2}}\left[\int_{-x}^{x}|\mathcal{F}^{-1}(r^+)(x,y)|^2 dy\right]^{\frac{1}{2}}dx$$
$$\leqq\int_1^\infty (2x)^{\frac{1}{2}}\left[\int_{-\infty}^{\infty}|\mathcal{F}^{-1}(r^+)(x,y)|^2 dy\right]^{\frac{1}{2}}dx$$
$$=\int_1^\infty (2x)^{\frac{1}{2}}\left[\int_{-\infty}^{\infty}|u(x,\eta)|^2 d\eta\right]^{\frac{1}{2}}dx,$$

where $u(x,\eta)=\int_{-\infty}^{\infty} r^+(\xi,\eta)e^{2\pi i x\xi}d\xi$.

The hypotheses of the theorem imply that $xf\in L^1(R^2)$ and $f\in L^1(R^2)$, so $\hat{f}$ and $\frac{\partial}{\partial\xi}\hat{f}\in C_0(R^2)$, that is, r and $r'\in C_0(R^2)$. Then by lemma 2

$$|u(x,\eta)|\leqq\frac{3}{8\pi^2x^2}\int_R\left|\frac{\partial^2}{\partial\xi^2}r(\xi,\eta)\right|d\xi.$$

So

$$\int_1^\infty\int_{-x}^{x}|\mathcal{F}^{-1}(r^+)(x,y)|dydx$$
$$\leqq\int_1^\infty\frac{3\sqrt{2}}{8\pi^2}x^{-\frac{3}{2}}\left[\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\left|\frac{\partial^2}{\partial\xi^2}r(\xi,\eta)\right|d\xi\right\}^2 d\eta\right]^{\frac{1}{2}}dx$$
$$\leqq\frac{3\sqrt{2}}{4\pi^2}\left[\int_{-\infty}^{\infty}\left(\frac{1}{2}\right)d\eta\int_{-\infty}^{\infty}\left|1+\pi^2\xi^2\right|\left|\frac{\partial^2}{\partial\xi^2}r(\xi,\eta)\right|^2 d\xi\right]^{\frac{1}{2}}$$

by the Cauchy-Schwarz inequality and

$$\int_{-\infty}^{\infty}|1+4\pi^2\xi^2|^{-1}d\xi=\frac{1}{2}.$$

Thus using the Plancherel theorem

$$\int_1^\infty\int_{-x}^{x}|\mathcal{F}^{-1}(r^+)(x,y)|dydx$$
$$\leqq\frac{3}{4\pi^2}\left[\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{\infty}\left|1+4\pi^2\xi^2\right|\left|\frac{\partial^2}{\partial\xi^2}\hat{f}(\xi,\eta)\right|^2 d\xi\right]^{\frac{1}{2}}$$
$$=\frac{3}{4\pi^2}\left[\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{\infty}|(1+2\pi i\xi)\frac{\partial^2}{\partial\xi^2}\hat{f}(\xi,\eta)|^2 d\xi\right]^{\frac{1}{2}}$$
$$=3\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1+\frac{\partial}{\partial x}\right)x^2 f(x,y)\,dydx\right]^{\frac{1}{2}}$$

as required. (Q. E. D.)

Remark 1. In proposition 1 and proposition 3 rapidly decreasing functions satisfies the hypothesis. But we should note that it is milder condition in growth and regularity than that of rapidly decreasing functions.

Remark 2. Here we consider only the positive part r^+ *of* $\hat{f}(\xi)=[r^+(\xi)-r^-(\xi)]+i[s^+(\xi)-s^-(\xi)]$. But other three parts can be considered by the same criteria.

Remark 3. For the case $n\geqq3$, a useful criterion did not be presented here. If $f\in L^1(R^n)$ and f is radial then $\hat{f}$ is continuously differentiable on $R^n\setminus\{0\}$; r^+ will be continuously differentiable only if $r\geqq0$ or if the zeroes of r are of order$\geqq2$ and so, in general, $\mathcal{F}^{-1}(r^+)\notin L^1(R^n)$. ([3], p. 35).

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