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An Exact Triangle induced by a Short Exact Sequence which Splits.

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0. Introduction

Let ${}_RM$ be the category of R-modules and R-homomorphisms, where R is a commutative ing with unity. Also let *COMP* be the category of all chain complexes and chain transforations. The differential modules in ${}_RM$ and some properties of them have been already introuced [4]. Also we obtained the exact triangle of homology modules of differential modules 5]. Using this facts, we proved that there exists the exact triangle of homology modules of ifferential modules induced by chain complexes C, D and E, that is, a short exact sequence $\longrightarrow C\longrightarrow D\longrightarrow E\longrightarrow 0$ in *COMP*. [Proposition 4].

The main object of this paper is to construct the exact triangle of direct sums of torsio. roducts induced by a short exact sequence $0\longrightarrow A'\longrightarrow A\longrightarrow A''\longrightarrow 0$ in ${}_RM$ which splitss Theorem 8].

Most of notations are taken from [2].

1. Preliminaries

Definition 1. Let X be an R-module and (C, ∂) a chain complex:

$$C: \cdots\longrightarrow C_{n+1}\xrightarrow{\partial} C_n\xrightarrow{\partial} C_{n-1}\longrightarrow\cdots$$

Then C is called *projective resolution* of X if

(1) $C_{-1}=X$

(2) $C_n=0$ for every $n<-1$

(3) C_n is a projective R-module for every $n\geq 0$.

Let X and Y be arbitrarily given R-modules. Select any projective resolution C of the module X

$$C: \cdots\longrightarrow C_{n+1}\xrightarrow{\partial} C_n\xrightarrow{\partial} C_{n-1}\longrightarrow\cdots$$

and consider its tensor product $C\otimes Y$ which is the sequence

$$C\otimes Y: \cdots\longrightarrow C_{n+1}\otimes Y\xrightarrow{\partial*} C_n\otimes Y\xrightarrow{\partial*} C_{n-1}\otimes Y\longrightarrow\cdots$$

where $\partial*$ stands for the tensor product $\partial\otimes i$ of the homomorphism ∂ and the identity endomorphism i of the module Y. Since $\partial*\circ\partial*=(\partial\otimes i)\circ(\partial\otimes i)=(\partial\circ\partial)\otimes(i\circ i)=0\otimes i=0$, $C\otimes Y$ is a semi-exact sequence and so $C\otimes Y$ is a chain complex. Thus for every integer n, the n-dimensional homology module $H_n(C\otimes Y)$ of $C\otimes Y$ is defined.

Proposition 2. *For any two projective resolution* C, D *of the module* X,

$$C: \cdots\longrightarrow C_{n+1}\xrightarrow{\partial} C_n\xrightarrow{\partial} C_{n-1}\longrightarrow\cdots$$
$$D: \cdots\longrightarrow D_{n+1}\xrightarrow{\delta} D_n\xrightarrow{\delta} D_{n-1}\longrightarrow\cdots$$

we have $H_n(C\otimes Y)\approx H_n(D\otimes Y)$ *for every integer* n.

Proof. The proof may be found in [2, p. 131].

The above module $H_n(C\otimes Y)$ depends essentially only on the integer n and given R-modules X, Y. Thus we make the following definition.

Definition 3. For every integer n, the R-module $H_n(C\otimes Y)$ is said to be the *n-dimensional torsion product* of the given modules X, Y and is denoted by the symbol $Tor_n^R(X, Y)$.

Proposition 4. *Let*

$$0\longrightarrow C\longrightarrow D\longrightarrow E\longrightarrow 0$$

be a given short exact sequence in COMP. Then there exists an exact triangle of homology modules of differential modules induced by chain complexes C, D *and* E:

$$\begin{array}{ccc} H(X) & \longrightarrow & H(Y) \\ & \nwarrow \quad \swarrow & \\ & H(Z) & \end{array}$$

where $X=\bigoplus_{n\in Z} C_n$, $Y=\bigoplus_{n\in Z} D_n$ *and* $Z=\bigoplus_{n\in Z} E_n$.

Proof. It follows from [3].

Proposition 5. *If a short exact sequence*

$$0\longrightarrow A'\longrightarrow A\longrightarrow A''\longrightarrow 0 \text{ in } {}_RM$$

splits, then for any R-module K, *the sequence*

$$0\longrightarrow A'\otimes K\longrightarrow A\otimes K\longrightarrow A''\otimes K\longrightarrow 0 \text{ in } {}_RM$$

is exact.

Proof. It follows from [2, p. 66].

Proposition 6. (3×3 *lemma*) *Consider the commutative diagram in* ${}_RM$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If the columns are exact and if the bottom two rows are exact, then the top row is exact.

Proof. It follows from [1, p. 175].

Proposition 7. *Consider the diagram in* ${}_RM$:

$$\begin{array}{ccccccccc}
 & & \downarrow & & & & \downarrow & & \\
 & & P_1' & & & & P_1'' & & \\
 & & d_1' \downarrow & & & & \downarrow d_1'' & & \\
 & & P_0' & & & & P_0'' & & \\
 & & \varepsilon' \downarrow & & & & \downarrow \varepsilon'' & & \\
0 & \longrightarrow & A' & \xrightarrow[i]{} & A & \xrightarrow[p]{} & A'' & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

where the columns are projective resolutions and the row is exact. Then there exists a pro-tive resolution of A and chain transformations so that the columns form an exact sequence COMP.

Proof. By induction, it sufficies to complete the 3×3 diagram

$$\begin{array}{ccccccc} & & 0 & & & 0 & \\ & & \downarrow & & & \downarrow & \\ & & K_0' & & & K_0'' & \\ & & \downarrow & & & \downarrow & \\ & & P_0' & & & P_0'' & \\ & & \varepsilon'\downarrow & & & \downarrow\varepsilon'' & \\ 0 & \longrightarrow & A' & \xrightarrow[i]{} A & \xrightarrow[p]{} & A'' & \longrightarrow 0 \\ & & \downarrow & & & \downarrow & \\ & & 0 & & & 0 & \end{array}$$

where the rows and columns are exact and P_0', P_0'' are projective. Define $P_0=P_0'\oplus P_0''$, $: P_0'\longrightarrow P_0$ *by* $x'\longrightarrow(x',0)$, *and* $p_0: P_0\longrightarrow P_0''$ *by* $(x',x'')\rightarrow x''$.

$$\begin{array}{ccccccc} P_0' & \xrightarrow{i_0} & P_0 & \xrightarrow{p_0} & P_0'' & & \\ \downarrow\varepsilon' & & \varepsilon\downarrow & & \downarrow\varepsilon'' & & \\ A' & \xrightarrow[i]{} & A & \xrightarrow[p]{} & A'' & \longrightarrow & 0 \end{array}$$

is clear that P_0 is projective and that

$$0\longrightarrow P_0'\xrightarrow[i_0]{} P_0\xrightarrow[p_0]{} P_0''\longrightarrow 0$$

exact. Since P_0'' is projective, there is a homomorphism $\sigma: P_0''\longrightarrow A$ with $p\circ a=\varepsilon''$ Define $: P_0\longrightarrow A$ by $\varepsilon: (x',x'')\longrightarrow(i\circ\varepsilon')(x')+\sigma(x'')$. It is an verification that, if $K_0=ker\,\varepsilon$, the sulting 3×3 diagram commutes. Exactness of the top is the 3×3 Lemma.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_0' & \longrightarrow & K_0=ker\,\varepsilon & \longrightarrow & K_0'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_0' & \xrightarrow[i_0]{} & P_0=P_0'\oplus P_0'' & \xrightarrow[p_0]{} & P_0'' & \longrightarrow & 0 \\ & & \varepsilon'\downarrow & & \downarrow\varepsilon & & \downarrow\varepsilon'' & & \\ 0 & \longrightarrow & A' & \xrightarrow[i]{} & A & \xrightarrow[p]{} & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

2. Main theorem

Theorem 8. *Let*

$$0\longrightarrow A'\longrightarrow A\longrightarrow A''\longrightarrow 0 \text{ in } {}_RM$$

e a given short exact sequence which splits. Then there exists an exact triangle of direct sums f torsion products, that is,

$$\begin{array}{ccc} \bigoplus_{n\in Z} Tor_n^R(A',K) & \longrightarrow & \bigoplus_{n\in Z} Tor_n^R(A,K) \\ \nwarrow & & \swarrow \\ & \bigoplus_{n\in Z} Tor_n^R(A'',K) & \end{array}$$

Proof. By Proposition 7, there exists an exact sequence

$$0\longrightarrow C\longrightarrow D\longrightarrow E\longrightarrow 0 \text{ in } COMP,$$

where C, D and E are projective resolutions A', A and A'' respectively. This means that, for every integer $n \in Z$,

$$0 \longrightarrow C_n \longrightarrow D_n \longrightarrow E_n \longrightarrow 0$$

is exact in ${}_RM$.

By Proposition 5

$$0 \longrightarrow C_n \otimes K \longrightarrow D_n \otimes K \longrightarrow E_n \otimes K \longrightarrow 0$$

is exact in ${}_RM$. Thus

$$0 \longrightarrow C \otimes K \longrightarrow D_n \otimes K \longrightarrow E \otimes K \longrightarrow 0$$

is exact in *COMP*. By Proposition 4, there exists an exact triangle of homology modules of differential modules induced by chain complexes $C \otimes K$, $D \otimes K$ and $E \otimes K$:

$$\begin{array}{ccc} \bigoplus_{n \in Z}(C_n \otimes K)) & \longrightarrow & H(\bigoplus_{n \in Z}(D_n \otimes K)) \\ \nwarrow & & \swarrow \\ & H(\bigoplus_{n \in Z}(E_n \otimes K)) & \end{array}$$

Also we obtain $H(\bigoplus_{n \in Z}(C_n \otimes K)) = \bigoplus_{n \in Z} H_n(C \otimes K) = \bigoplus_{n \in Z} Tor_n{}^R(A', K)$. Similarly $H(\bigoplus_{n \in Z}(D_n \otimes K)) = \bigoplus_{n \in Z} Tor_n{}^R(A, K)$ and $H(\bigoplus_{n \in Z}(E_n \otimes K)) = \bigoplus_{n \in Z} Tor_n{}^R(A'', K)$.

This proof is complet.

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