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Some Properties of Homomorphisms in BCK-algebras

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I. Introduction and Preliminaries

In [3], K. Iseki proved the following theorem:

Theorem 1.1. *Let X, Y, Z be BCK-algebras, and let $h: X\to Y$ be an epimorphism, and let* $: X\to Z$ *be a homomorphism. If $Ker(h) \subset Ker(g)$, then there is a unique homomorphism $f : Y\to Z$* *tisfying $f\circ h=g$.*

In this paper we will investigate some properties on homomorphisms in BCK-algebras.

First of all, we recall the notion of BCK-algebras ([2], [3]).

Definition 1.2 Let X be a set with a binary operation $*$ and a constant 0. X is called a *CK-algebra,* if it satisfies the following conditions:

(1) $(x*y)*(x*z)\leq z*y$

(2) $x*(x*y)\leq y$

(3) $x\leq x$

(4) $0\leq x$

(5) $x\leq y$, $y\leq x$ implies $x=y$.

here $x\leq y$ is defined by $x*y=0$

A BCK-algebra has the following basic properties (for the proof, see [1], [3]):

(6) X is a partially ordered set with respect to $\leq$,

(7) $x\leq y$ implies $x*z\leq y$, and $z*y\leq z*x$,

(8) $(x*y)*z=(x*z)*y$,

(9) $(x*y)*(z*y)\leq x*z$,

(10) $x*y\leq z$ implies $x*z\leq y$,

(11) $x*y\leq x$,

(12) $x*0=x$.

Definition 1.3. Let X, X' be BCK-algebras. A mapping $f:X\to X'$, is called a *homomorphism* f, for every pair of x, y *of* X,

$$f(x*y)=f(x)*f(y).$$

'or a homomorphism f,

$$f(0)=f(0*0)=f(0)*f(0)=0$$

Hence $f(0)=0$.

The kernel of f, $Ker(f)$, and the image of f, $Im(f)$, are defined by

$$Ker(f)=\{x \mid f(x)=0\},$$
$$Im(f)=\{f(x)\in X' \mid x\in X\}.$$

If a homomorphism $f: X\to X'$ is onto $(1-1)$, then it is called an *epimorphism* (*monomorphism*).

II. Theorems

Theorem 2.1. *Let $f: X\to Y$ be a homomorphism of BCK-algebras. Then f is monomorphism if and only if $Ker(f)=0$.*

Proof. Let $x\in Ker(f)$. Then $f(x)=0$. Since $f(0)=0$, and since f is a monomorphism, $x=0$. Conversely suppose $f(x_1)=f(x_2)$ for all $x_1, x_2\in X$. Then $f(x_1*x_2)=f(x_1)*f(x_2)=0$. Hence $x_1*x_2\in Ker(f)$, and so $x_1*x_2=0$. Therefore we have $x_1\le x_2$. Similarly, we obtain $x_2\le x_1$. Hence, by (5), $x_1=x_2$. This shows that f is a monomorphism.

Theorem 2.2. *In Theorem 1.1, we have*

(a) $Ker(f)=h(Ker(g))$

(b) $Im(f)=Im(g)$

(c) *f is a monomorphism iff $Ker(h)=Ker(g)$*

(d) *f is an epimorphism iff g is an epimorphism.*

Proof. (a) If $y\in h(Ker(g))$, then there exist an element x in $Ker(g)$ such that $h(x)=y$. Then $f(y)=f(h(x))=g(x)=0$. Hence $y\in Ker(f)$, *i.e.* $h(Ker(g))\subset Ker(f)$. Conversely, let $y\in Ker(f)$. Since $Ker(f)\subset Y$, for this y, there exists an element $x\in X$ such that $h(x)=y$. Then $0=f(y)=f(h(x))=g(x)$, and so $x\in Ker(g)$. Hence $y=h(x)\in h(Ker(g))$.

Therefore $Ker(f)\subset h(Ker(g))$.

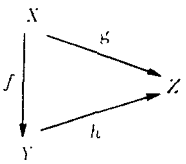
(b) Let $z\in Im(f)$. Then there exist $y\in Y$ such that $f(y)=z$. Since h is an epimorphism, for this y, there exists $x\in X$ such that $h(x)=y$. Hence $z=f(y)=f(h(x))=g(x)\in Im(g)$, *i.e.*, $Im(f)\subset Im(g)$. The reverse inclusion is obvious.

(c) This follows from (a).

(d) Since $g=f\circ h$, we have (d).

Theorem 2.3. *Let X, Y, Z be BCK-algebras, and let $g: X\to Z$ be a homomorphism; and let $h: Y\to Z$ be a monomorphism with $Im(g)\subset Im(h)$. Then there is a unique homomorphism $f: X\to Y$ satisfying $g=h\circ f$.*

Proof. For each $x\in X$, $g(x)\in Im(g)\subset Im(h)$. Since h is a monomorphism, there exists a unique $y\in Y$ such that $h(y)=g(x)$. Therefore there is a function $f: X\to Y$, $x\to y$, such that $h\circ f=g$. To show that f is a homomorphism, let $x_1, x_2\in X$, then $g(x_1*x_2)=h(f(x_1*x_2))$. On the other hand, since g is a homomorphism,



$$g(x_1*x_2)=g(x_1)*g(x_2)=h(f(x_1))*h(f(x_2))=h(f(x_1)*f(x_2)).$$

Hence $h(f(x_1*x_2))=h(f(x_1)*f(x_2))$. Since h is a monomorphism, $f(x_1*x_2)=f(x_1)*f(x_2)$. The uniqueness of f is trivial since h is a monomorphism.

Theorem 2.4. *In Theorem 2.3, we have*

(a) $Ker(f)=Ker(g)$,

(b) $Im(f)=h^{-1}(Im(g))$,

(c) *f is a monomorphism, iff g is a monomorphism*

(d) *f is an epimorphism iff $Im(h)=Im(g)$.*

Proof. (a) Clear.

(b) If $y \in h^{-1}(\mathrm{Im}(g))$, then $h(y) \in Im(g)$. So there exists an element $x \in X$ such that $g(x)=h(y)$. Since $g=h\circ f$, and since h is a monomorphism, $f(x)=y$. Hence $y \in Im(f)$. This shows $h^{-1}(Im(g)) \subset Im(f)$. The reverse inclusion is clear.

(c) Since $g=h\circ f$, we have (c).

(d) ($\Longrightarrow$) Suppose f is an epimorphism. Then, for any $y \in Y$ there exists an element $x \in X$ such that $f(x)=y$. Hence $h(y)=h(f(x))=g(x) \in Im(g)$. This shows $Im(h) \subset Im(g)$. If $z \in Im(g)$, then there is $x \in X$ such that $g(x)=z$. Since $g=h\circ f$, $z \in Im(h)$. This shows $Im(g) \subset Im(h)$.

($\Longleftarrow$) Clear.

Corollary 2.5. *Let X and K be BCK-algebras and let $j: K \to X$ be a monomorphism with $Im(j)=I$. Then there is a unique isomorphism $k: I \to K$ such that $j\circ k=i_I$ (inclusion map).*

Proof. Let $I=X$, $X=Z$, $K=Y$, $i_I=g$ *and* $j=h$ in Theorem 2.3. Then we have the corollary by (c) and (d) of Theorem 2.4.

References

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