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— Research Paper —

The Capacitated Solid Transportion Problem (Planar Constraints) with Upper and Lower Bounds on Rim Conditions

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Abstract

In the present paper, the capacitated solid transportation problem (planar constraints) is considered with upper and lower bounds on the rim conditions. The considered model has been shown to be equivalent to a standard capacitated 3-index transportation problem. A procedure to solve any capacitated 3-index transportion problem is also suggested. Some special cases of the considered model have been shown to conform to possible practical situations.

1. Introduction

The solid transportation problem considered by Haley [5, 6] is the problem of minimizing the total transportation cost involved in shipping various commodities from a number of warehouses to number of markets. He has termed his constraints as 'planar sums' or 'planar constraints'.

An important limitation of the 'planar constraints' is that the availability of each commodity at each warehouse, the demand of each commodity at the markets and the total supply from a particular warehouse to a market are fixed quantities. Furthermore no restriction is imposed on the allocation of various commodities from a warehouse to a market.

It is quite natural with such rigidity in availabilites, demands and the total supply and unrestricted allocation, the problem goes away from reality and finds very little application.

A hypothetical situation, in which a manufacturer who just starts the production of hetrogeneous products in his newly established production units to distribute in the city markets, is not sure about the maximum output, as the production units are at their infancy. At the same time he is ignorant about the maximum or the minimum demand of a particular commodity at a specified market. The presently considered model highlights the above situations and also many other complex varieties.

The classical transportation problem with upper and lower bounds on the rim conditions have

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been studied by Charnes, Glover and Klingman [2, 3, 4]. In this paper, the capacitated solid transportation problem with upper and lower bounds on the rim conditions (*i.e.* availability of each product at the warehouses, the demand of each product at the markets and the total supply from each warehouse to a market), is considered. These rim conditions are the 'planar sums' as defined by Haley [5].

The problem is transformed into a capacitated solid transportation problem with equality type of constraints and some of decision variables bounded above by the difference of maximum and minimum availabilities of a product at a warehouse, demands at the markets and the supplies from a warehouse to a market.

This transformation is brought about by the introduction of a dummy warehouse, a dummy commodity and a dummy market.

The equivalence between the original and the transformed problem is estblished by a theorem. A solution method is suggested for any problem of the form of the transformed problem, so that a solution of the transformed problem thus obtained would give the optimal solution of the original problem.

Three special applications are discussed at the end.

2. Theoretical Development

The capacitated solid transportation problem (planar constraints) with lower and upper bounds on the rim conditions is defined as:

(P-1): Minimize $Z = \sum_I \sum_J \sum_K C_{ijk} \ x_{ijk}$

s. t.

$$a_{jk} \leqq \sum_I x_{ijk} \leqq A_{jk}, \ j \in J, \ k \in K$$

$$b_{ik} \leqq \sum_J x_{ijk} \leqq B_{ik}, \ i \in I, \ k \in K$$

$$e_{ij} \leqq \sum_K x_{ijk} \leqq E_{ij}, \ i \in I, \ j \in J$$

and $$0 \leqq x_{ijk} \leqq u_{ijk}, \ i \in I, \ j \in J, \ k \in K$$

where

$I = \{1, 2, \ldots, m\}$, the set of warehouses

$J = \{1, 2, \ldots, n\}$, the set of markets

$K = \{1, 2, \ldots, p\}$, the set of commodities

x_{ijk} = amount of k^{th} commodity shipped from warehouse i to market j.

c_{ijk} = cost per unit of the k^{th} commodity shipped from warehouse i to market j.

a_{jk} = minimum demand of the k^{th} product at the j^{th} market.

A_{jk} = maximum demand of the k^{th} product at the j^{th} market.

b_{ik} = minimum availability of the k^{th} product at the i^{th} warehouse.

B^{ik} = maximum availability of the k^{th} product at the i^{th} warehouse.

e_{ij} = minimum supply from the i^{th} warehouse to the j^{th} market.

E_{ij} = maximum supply from the i^{th} warehouse to the j^{th} market.

For feasibility, it is assumed that

$$A_{jk} \geqq a_{jk} \geqq 0, \quad j \in J,\ k \in K$$
$$B_{ik} \geqq b_{ik} \geqq 0, \quad i \in I,\ k \in K$$
$$E_{ij} \geqq e_{ij} \geqq 0, \quad i \in I,\ j \in J$$

$$\sum_J e_{ij} \leqq \sum_K B_{ik}, \ i \in I$$
$$\sum_I e_{ij} \leqq \sum_K A_{jk}, \ j \in J$$
$$\sum_I b_{ik} \leqq \sum_J A_{jk}, \ k \in K$$
$$\sum_I \sum_K b_{ik} \geqq \sum_J \sum_K A_{jk}$$
$$\sum_I \sum_J e_{ij} \leqq \sum_I \sum_K B_{ik}$$

The above problem is equivalent to the following standard three-index capacitated transportation problem, termed as related problem (RP).

(RP) : Minimize $\sum_{I'} \sum_{J'} \sum_{K'} c_{ijk}\ x_{ijk}$

s. t.

$$\sum_{I'} x_{ijk} = A_{jk}\ ,\ j \in J,\ k \in K$$
$$\sum_{J'} x_{ijk} = B_{ik}\ ,\ i \in I,\ k \in K$$
$$\sum_{K'} x_{ijk} = E_{ij}\ ,\ i \in I,\ j \in J$$
$$\sum_{I'} x_{ij\ p+1} = A_{j\ p+1} = \sum_I E_{ij}$$
$$\sum_{I'} x_{i\ n+1\ k} = A_{n+1\ k} = \sum_I B_{ik}$$
$$\sum_{I'} x_{i\ n+1\ p+1} = A_{n+1\ p+1} = R$$
$$\sum_{J'} x_{ij\ p+1} = B_{i\ p+1} = \sum_J E_{ij}$$
$$\sum_J x_{m+1\ jk} = B_{m+1\ k} = \sum_J A_{ik}$$
$$\sum_{J'} x_{m+1\ j\ p+1} = B_{m+1\ p+1} = R$$
$$\sum_{K'} x_{m+1\ jk} = E_{m+1\ j} = \sum_K A_{jk}$$
$$\sum_{K'} x_{i\ n+1\ k} = E_{i\ n+1} = \sum_K B_{ik}$$
$$\sum_{K'} x_{m+1\ n+1\ k} = E_{m+1\ n+1} = R,$$

where R is an arbitrary large number.

$$0 \leqq x_{ijk} \leqq u_{ijk}\ ,\ i \in I,\ j \in J,\ k \in K$$
$$0 \leqq x_{m+1\ jk} \leqq A_{jk} - a_{jk}\ ,\ j \in J,\ k \in K$$
$$0 \leqq x_{i\ n+1\ k} \leqq B_{ik} - b_{ik}\ ,\ i \in I,\ k \in K$$
$$0 \leqq x_{ij\ p+1} \leqq E_{ij} - e_{ij}\ ,\ i \in I,\ j \in J$$
$$0 \leqq x_{i\ n+1\ p+1} \leqq \min(E_{i\ n+1},\ B_{i\ p+1}),\ i \in I$$

$$0 \leqq x_{m+1\ j\ p+1} \leqq \min(A_{j\ p+1},\ E_{m+1\ j}),\ j \in J$$
$$0 \leqq x_{m+1\ n+1\ k} \leqq \min(A_{n+1\ k},\ B_{m+1\ k}),\ k \in K$$
$$0 \leqq x_{m+1\ n+1\ p+1} \leqq R.$$

$$c_{ij\ p+1} = 0,\ i \in I,\ j \in J \qquad c_{i\ n+1\ p+1} = 0,\ i \in I$$
$$c_{i\ n+1\ k} = 0,\ i \in I,\ k \in K \qquad c_{m+1\ j\ p+1} = 0,\ j \in J$$
$$c_{m+1\ jk} = 0,\ j \in J,\ k \in K \qquad c_{m+1\ n+1\ k} = 0,\ k \in K$$
$$c_{m+1\ n+1\ p+1} = 0.$$

Theorem I. The solution $\{x_{ijk} = x^*_{ijk}\}, i \in I,\ j \in J,\ k \in K$ is feasible (optimal) for problem (P-1), if and only if, the solution $\{x_{ijk}\},\ i \in I',\ j \in J',\ k \in K'$
where

$$x_{ijk} = x^*_{ijk},\ i \in I,\ j \in J,\ k \in K$$
$$x_{i\ n+1\ k} = B_{ik} - \sum_J x^*_{ijk},\ i \in I,\ k \in K$$
$$x_{m+1\ jk} = A_{jk} - \sum_I x^*_{ijk},\ j \in J,\ k \in K$$
$$x_{ij\ p+1} = E_{ij} - \sum_K x^*_{ijk},\ i \in I,\ j \in J$$
$$x_{i\ n+1\ p+1} = \sum_J \sum_K x^*_{ijk},\ i \in I$$
$$x_{m+1\ j\ p+1} = \sum_I \sum_K x^*_{ijk},\ k \in K,\quad j \in J$$
$$x_{m+1\ n+1\ k} = \sum_I \sum_J x^*_{ijk},\quad k \in K$$
$$x_{m+1\ n+1\ p+1} = R - \sum_I \sum_J \sum_K x^*_{ijk},$$ is feasible (optimal) for (RP).

Proof. Let $\{x_{ijk} = x^*_{ijk}\},\ i \in I,\ j \in J,\ k \in K$ be a feasible solution of (P-1).

For $j \in J,\ k \in K$

$$\sum_{I'} x_{ijk} = \sum_I x_{ijk} + x_{m+1\ jk}$$
$$= \sum_I x^*_{ijk} + (A_{jk} - \sum_I x^*_{ijk}) \text{ (by def.)}$$
$$= A_{ik}$$

Hence $\sum_{I'} x_{ijk} = A_{jk},\ j \in J,\ k \in K$

Similarly, it can be shown that

$$\sum_{J'} x_{ijk} = B_{ik},\ i \in I,\ k \in K$$
$$\sum_{K'} x_{ijk} = E_{ij},\ i \in I,\ j \in J \quad \cdots\cdots\cdots (1)$$

Consider $j \in J$. Then

$$\sum_I x_{ij\,p+1} = \sum_I x_{ij\,p+1} + x_{m+1\ j\ p+1}$$
$$= \sum_I (E_{ij} - \sum_K x^*_{ijk}) + \sum_I \sum_K x^*_{ijk} \text{ (by def.)}$$
$$= \sum_I E_{ij} = A_{j\ p+1}$$

Therefore $\sum_{I'} x_{ij\ p+1} = A_{j\ p+1},\ j \in J$

Similarly it follows that

$$\left.\begin{array}{l}
\sum_{I'} x_{i\,n+1\;k}=\sum_{I} B_{ik}=A_{n+1\,k},\; k\in K \\
\sum_{I'} x_{i\,n+1\;p+1}=R=A_{n+1\;p+1} \\
\sum_{J'} x_{ij\,p+1}=\sum_{J} E_{ij}=B_{i\;p+1} \\
\sum_{J'} x_{m+1\;jk}=\sum_{J} A_{jk}=B_{m+1\;k} \\
\sum_{J} x_{m+1\;j\;p+1}=R=B_{m+1\;p+1} \\
\sum_{K'} x_{i\;n+1\;k}=\sum_{K} B_{ik}=E_{i\;n+1} \\
\sum_{K'} x_{m+1\;n+1\;k}=R=E_{m+1\;n+1} \\
\sum_{K'} x_{m+1\;jk}=\underset{K}{E}\, A_{jk}=E_{m+1\;j}
\end{array}\right\} \quad \cdots\cdots\cdots (2)$$

Now $\{x^*_{ijk}\}$ $i\in I,\ j\in J,\ k\in K$ being a solution of $(P-1)$,

$x^*_{ijk}\leqq u_{ijk},\ i\in I,\ j\in J,\ k\in K$ ········· (3)

Now for $j\in J,\ k\in K$

$$x_{m+1\;jk}=A_{jk}-\sum_{I} x^*_{ijk} \qquad \text{by def.}$$

$$a_{jk}\leqq\sum_{I} x^*_{ijk}\leqq A_{jk},\quad \text{by the feasibility conditions of } (P-1)$$

$$\rightarrow x_{m+1\;jk}>0$$

and $x_{m+1\;jk}\leqq A_{jk}-a_{jk}$.

$$\left.\begin{array}{l}
\text{Hence } 0\leqq x_{m+1\;jk}\leqq A_{jk},\ j\in J,\ k\in K \\
\text{Similarly} \\
\qquad 0\leqq x_{i\;n+1\;k}\leqq B_{ik},\ i\in I,\ k\in K \\
\qquad 0\leqq x_{ij\;p+1}\leqq E_{ij}-e_{ij},\ i\in I,\ j\in J \\
\text{Now } x_{m+1\,n+1\,k}=\sum_{I}\sum_{J} x^*_{ijk}\leqq \sum B_{ik}=A_{n+1\,k},\ k\in K
\end{array}\right\} \quad \cdots\cdots\cdots (4)$$

$$\rightarrow x_{m+1\;n+1\;k}\leqq A_{n+1\;k},\ k\in K$$

Also $x_{m+1\;n+1\;k}=\sum_{I}\sum_{J}' x^*_{ijk}=\sum_{I}\sum_{I} x^*_{ijk}\leqq\sum_{J} A_{jk}=B_{m+1\;k}$, $k\in K$

$$\left.\begin{array}{l}
\text{Hence } x_{m+1\;n+1\;k}\leqq min(A_{n+1\;k},\ B_{m+1\,k},\ B_{m+1\;k}),\ k\in K \\
\text{Similarly } x_{m+1\;j\;p+1}\leqq min(A_{i\;p+1},\ E_{i\;n+1}),\ i\in I \\
\qquad\qquad x_{m+1\;j\;p+1}\leqq min(A_{i\;p+1},\ E_{m+1}),\ j\in J
\end{array}\right\} \quad \cdots\cdots\cdots (5)$$

Again $\sum_{K'} x_{m+1\,n+1\;k}=B_{m+1\;n+1}=R$

$$\rightarrow x_{m+1\;n+1\;p+1}\leqq R \quad \cdots\cdots\cdots (6)$$

(1), (2), (3), (4), (5), (6) together prove that

$\{x_{ijk}\}$, $i\in I'$, $j\in J'$, $k\in K'$ as defined in the theorem is a feasible solution of (RP).

The converse of the theorem is obvious.

It can be easily noted that the objective function of the problems ($P-1$) and RP at their respective coresponding solutions are same.

Hence, if one of the solutions is optimal, the other is also optimal.

3. Solution Procedure for Capacitated 3-index Problem with Planar Constraints

($m \times n \times p$) Capactated 3-index problem with planar constaints is :

(P) : Minimize $\sum_I \sum_J \sum_K c_{ijk}\, x_{ijk}$

s. t.

$$\sum_I x_{ijk} = A_{jk},\ j \in J,\ k \in K$$

$$\sum_J x_{ijk} = B_{ik},\ i \in I,\ k \in K$$

$$\sum_K x_{ijk} = E_{ij},\quad i \in I,\ j \in J$$

$$0 \leqq x_{ijk} \leqq u_{ijk}$$

where

$I = \{ 1, 2, \ldots, m \}$

$J = \{ 1, 2, \ldots, n \}$

$K = \{ 1, 2, \ldots, p \}$

The procedure suggested below is to formulate an equivalent uncacitated ($m \times n \times p \times 2$) 4-index problem, whose optimal solution yields an optimal solution of the above problem.

Now $x_{ijk} \leqq u_{ijk},\ i \in I,\ j \in J,\ k \in K$

$\rightarrow x_{ijk} + y_{ijk} = u_{ijk},\ y_{ijk} \geqq 0,\ i \in I,\ j \in J,\ k \in K$

This suggests that for each commodity k, the cell (i, j, k) can be further subdivided into two cells (i, j, k, 1) and (i, j, k, 2), with allocations x_{ijk1} and x_{ijk2}, s. t. $x_{ijk1} = x_{ijk}$, $x_{ijk2} = y_{ijk}$, where

$$\sum_{l=1}^{2} x_{ijkl} = u_{ijk}.$$

Define $c_{ijk1} = c_{ijk}$ and $c_{ijk2} = 0,\ i \in I,\ j \in J,\ k \in K.$

Consider the following ($m \times n \times p \times 2$) 4-index transportation problem :

(Q): Minimize $\sum_I \sum_J \sum_K \sum_L c_{ijkl}\, x_{ijkl}$

s. t.

$$\sum_I x_{ijkl} = A_{jkl},\ j \in J,\ k \in K,\ l \in L$$

$$\sum_J x_{ijkl} = B_{ikl},\ i \in I,\ k \in K,\ l \in L$$

$$\sum_K x_{ijkl} = E_{ijl},\quad i \in I,\ j \in J,\ l \in L$$

$$\sum_{L} x_{ijkl} = F_{ijk}, \ i \in I, \ j \in J, \ k \in K$$

$$x_{ijkl} \geqq 0, \ i \in I, \ j \in J, \ k \in K, \ l \in L$$

where $L = \{1, 2\}$

and
$$A_{jk1} = A_{jk}$$
$$A_{jk2} = \sum_{I} u_{ijk} - A_{jk}$$
$$B_{ik1} = B_{ki}$$
$$B_{ik2} = \sum_{J} u_{ijk} - E_{ij}$$
$$F_{ijk} = u_{ijk}, \ i \in I, \ j \in J, \ k \in K$$

The above problem (Q) is a balanced uncapacited 4-index problem, the solution method of which is given in [1].

Theorem II. A feasible solution of (P) provides a feasible solution of (Q) and conversely.

Proof. If $\{x_{ijk}\}, i \in I, j \in J, k \in K$ is a feasible solution of (P), then since $x_{ijk} \leqq u_{ijk}$, so setting $x_{ijk1} = x_{ijk}$, $x_{ijk2} = u_{ijk} - x_{ijk}$, it is quite obvious that $\{x_{ijkl}\}, i \in I, j \in J, k \in K, l \in L$ is a feasible solution of (Q).

Conversely, let $\{x_{ijkl}\}, i \in I, j \in J, k \in K, l \in L$ be a feasible solution of (Q).

Consider a solution $\{x_{ijk}\}, i \in I, j \in J, k \in K$

where $x_{ijk} = x_{ijk1}$.

Since $x_{ijk1} \geqq 0$, therefore $x_{ijk} \geqq 0$

$$\sum_{I} x_{ijk} = \sum_{I} x_{ijk1} = A_{jk1} = A_{jk}, \ j \in J, \ k \in K$$

$$\sum_{J} x_{ijk} = \sum_{J} x_{ijk1} = B_{ik1} = B_{ik}, \ i \in I, \ k \in K$$

$$\sum_{K} x_{ijk} = \sum_{K} x_{ijk1} = E_{ij1} = E_{ij}, \ i \in I, \ j \in J$$

and $\sum_{L} x_{ijk} = F_{ijk1} = u_{ijk}, \ i \in I, \ j \in J, \ k \in K$

$$\rightarrow x_{ijk1} \leqq u_{ijk}$$

$$\rightarrow x_{ijk} \leqq u_{ijk}, \ i \in I, \ j \in J, \ k \in K$$

Hence $\{x_{ijk}\}, i \in I, j \in J, k \in K$ is a feasible solution of (P).

Remark. Since $c_{ijk1} = c_{ijk}$ and $c_{ijk2} = 0$, $i \in I, j \in J, k \in K$ the values of the objective function corresponding to the fasible solutlons of (P) and (Q) are equal. It is easy to check that an optimal soltion of (Q) provides an optimal solution of (P) and coversely. Hence when (Q) is solved by the method given in [1], that solution will yield an optimal solution of (P), by the methed given in Theorem II.

The related problem (RP) being a $(m+1) \times (n+1) \times (p+1)$ 3-index capactated problem can be solved by the above procedure. The optimal solution thus obtained would yield an optimal solution of $(P-1)$, as proved in Theorem I.

4. Some Special Cases of the Present Model

1. Minimize $\sum_I \sum_J \sum_K c_{ijk} \ x_{ijk}$

s. t.

$$\sum_I x_{ijk} \leqq a_{jk}, \ j \in J, \ k \in K$$

$$\sum_J x_{ijk} \geqq b_{ik}, \ i \in I, \ k \in K$$

$$\sum_K x_{ijk} = e_{ij}, \ i \in I, \ j \in J$$

$$0 \leqq x_{ijk} \leqq u_{ijk}, \ i \in I, \ j \in J, \ k \in K$$

assume $c_{ijk} \geqq 0, \ i \in I, \ j \in J, \ k \in K$

and for feasibility

$$\sum_I e_{ij} \geqq \sum_K a_{jk}$$

$$\sum_J e_{ij} \geqq \sum_K b_{ik}$$

$$\sum_I \sum_J e_{ij} \geqq max \left(\sum_I \sum_K a_{jk}, \ \sum_I \sum_K b_{ik}\right)$$

Taking $A_{jk} = \sum_I e_{ij}, \ j \in J, \ k \in K$

$$B_{ik} = \sum_J e_{ij}, \ i \in I, \ k \in K$$

$$E_{ij} = e_{ij} \quad , \quad i \in I, \ j \in J$$

the above problem is of the same form as (P–1) and hence can be solved by related problem.

This model refers to the wartime distribution of various commodities to various military camps, where the maximum demand of the k^{th} commodity at the j^{th} camp is not known and also the maximum availability of the k^{th} commodity at the j^{th} camp is not known and also the maximum availability of the k^{th} commodity at the i^{th} production unit is unknown, for emergency units have been built up, whose production capacity is uncertain, but the total supply from i^{th} production unit to j^{th} military camp is specified and has to be maintained exactly.

2. Minimize $\sum_I \sum_J \sum_K c_{ijk} \ x_{ijk}$

s. t. $\sum_I x_{ijk} \geqq a_{jk}, \ j \in J, \ k \in K$

$$\sum_J x_{ijk} = b_{ik}, \ i \in I, \ k \in K$$

$$\sum_K x_{ijk} = e_{ij}, \ i \in I, \ j \in J$$

$$0 \leqq x_{ijk} \leqq u_{ijk}, \ i \in I, \ j \in J, \ k \in K$$

$$\sum_K b_{ik} = \sum_J e_{ij}, \ i \in I$$

$$\sum_I b_{ik} \geqq \sum_J a_{ik}, \ k \in K$$

$$\sum_I e_{ij} \geqq \sum_K a_{jk}, \ j \in J$$

This is also a special case of (P–1) and can be solved extending to the form of (RP).

This model refers to the case of a manufacturer, who has just started his business and is therefore uncertain of the maximum demand of the various commodities at the markets. He also has sound production units, so that production of the k^{th} commodity at the i^{th} unit is exactly known can exactly specify total supply cf all the commodities from any production unit to a market.

3. Minimize $\sum_I \sum_J \sum_K c_{ijk} \; x_{ijk}$

$$\text{s. t.} \quad \sum_I x_{ijk} \geqq a_{jk}, \; j \in J, \; k \in K$$

$$\sum_J x_{ijk} \geqq b_{ik}, \; i \in I, \; k \in K$$

$$\sum_K x_{ijk} \geqq e_{ij}, \quad i \in I, \; j \in J$$

$$0 \leqq x_{ijk} \leqq u_{ijk}, \; i \in I, \; j \in J, \; k \in K$$

The above problem becomes a special case of (P–1), if we assume

$$A_{jk} = \max\,(a_{jk}, \; \sum_I b_{ik}, \; \sum_I e_{ij})$$

$$B_{ik} = \max\,(b_{ik}, \; \sum_J a_{jk}, \; \sum_J e_{ij})$$

$$E_{ij} = \max\,(e_{ij}, \; \sum_K a_{jk}, \; \sum_K b_{ik}).$$

This case is applicable to the situations, where the manufacturer wants to save the goodwill loss by meeting the minimum demand, supplying at the minimum level, maintaining the minimum level of production.

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