Journal of the Korean
OR/MS Society
Vol. 10, No. 2, Oct. 1985

— Research Paper —

Joint Optimization of Redundancy and Component-Reliabilities in a Series System

Bong Jin Yum*

Abstract

Systems reliability can be improved by using redundancy and/or developing more reliable components. This paper considers a joint optimization of both alternatives for a series system. It is shown that the n-stage optimization problem can be decomposed into n single stage subproblems. Each subproblem is further transformed into a univariate optimization problem for which a simple and efficient solution method is developed.

1. Introduction

Basically, there are two ways of achieving high system reliability. One is to use redundancy and the other is to use more reliable components. Many authors ([2], [3], [5], −[7], [9]) proposed various models and solution methods for a joint optimization of redundancy and component-reliabilities. This paper considers the similar problem for a series system with n independent stages (or subsystems). It is shown that the n-stage reliability and redundancy optimization problem can be decomposed into n single stage subproblems for which solutions can be derived more simply and efficiently. The overall optimization can be accomplished by dynamic programming or other search techniques.

In passing, it is worth noting that Tillman *et al.* [10] thoroughly reviewed optimization techniques for systems reliability when redundancy is of primary concern.

2. Notation

n	Number of stages in series.
r_i	Reliability of each component in stage i, $0 < r_i < 1$.
x_i	Total number of redundant components in stage i.
$C_i(r_i)$	Cost of each component with reliability r_i in stage i.
R_s	Lower bound of system reliability.
R_i	Reliability of stage i.
Q_i	$1 - R_i$.
R	$(R_1, R_2, \ldots, R_n)$.
Σ_i, Π_i	Sum or Product over the domain of index.

*Dept. of Industrial Engineering, KAIST

3. The Model and Decomposition

A reliabilty and redundancy optimization problem for a n-stage series system is given by

Problem I :

$$\text{Minimize} \quad \sum_i x_i \; C_i(r_i) \qquad (1)$$

$$\text{subject to} \quad \Pi_i \{ 1-(1-r_i)^{x_i} \} \geqq R_s \qquad (2)$$

$$x_i \geqq 1, \text{ integers for all } i$$

where the component reliability r_i satisfies $0 < r_i < 1$, x_i is the number of independent, identical components at stage i, and R_s $(0 < R_s < 1)$ is given. It is assumed that each cost function $C(r)$ is strictly increasing with repsect to r and is continuously differentiable. Table 1 shows some cost functions which appear in the literature. The reason why $r_i = 0$ or 1 is not considered is as follows. If $r_i = 0$, then the system reliability becomes 0, and therefore, Problem I becomes infeasible under the assumption that $R_s > 0$. Further, stage i with $r_i = 1$ can be eliminated without loss of generality since optimal value of x_i equals 1 and the stage reliability becomes 1.

Note that Problem I is equivalent to

Problem I-A:

$$\text{Minimize} \quad \sum_i x_i \; C_i(r_i)$$

$$\text{subject to} \quad 1-(1-r_i)^{x_i} = R_i \text{ for all } i \qquad (3)$$

$$\Pi_i R_i = R_s \qquad (4)$$

$$x_i \geqq 1, \text{ integers for all } i.$$

The equality in Eq. (3) or (4) is justified since the objective function is assumed to be strictly increasing with respect to r_i's.

Table 1. Examples of Cost Functions.

Author	C(r)	C(r)/C′(r)
Aggarwal, Gupta [1]	$k[\tan(\pi r/2)]^a$ $k > 0,\ 1 \leqq a \leqq 2$	$\sin(\pi r)/\pi a$
Tillman, et al. [9]	$k(-\ln r)^a$ $k > 0,\ a > 1$	$(r \ln r)/a$
Misra, Ljubojevic [5]	$k \exp[a/(1-r)]$ $k > 0,\ a > 0$	$(1-r)^2/a$
Tillman, et al. [8]	kr^a $k > 0$	r/a

Suppose $R = (R_1, R_2, \ldots, R_n)$ which satisfies Eq. (4) is given. Then, Problem I-A can be decomposed into n single-stage subproblems in each of which optimal x_i and r_i must be determined. Many techniques are conceivable to optimally determine R. In this paper the dynamic

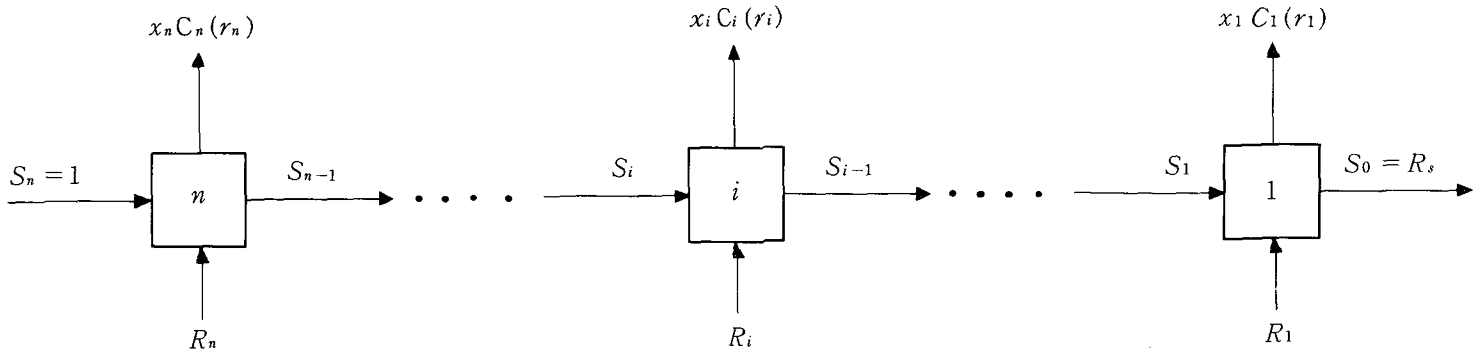

Fig. 1: Stagewise Representation of a Series System.

programming approach is presented due to its simplicity. Other search techniques may be also employed.

In Figure 1, subsystem i corresponds to stage i where

decision variable $= R_i$, $R_s < R_i < 1$,

input state variable $= S_i = \prod_{k=i+1}^{n} R_k$,

state transition: $S_{i-1} = S_i R_i$, and

return function $= x_i C_i(r_i)$.

Then, a backward recursion procedure may be described as follows.

At stage 1,

$$f_1(S_1) = \min_{R1} \{x_1 C_1(r_1): 1-(1-r_1)^{x_1} = R_1, \quad S_1 R_1 = S_0 = R_s\}. \quad (5)$$

Since $S_1 R_1 = R_s$, there is no optimization at stage 1 and $R_1^* = R_s/S_1$.

Then, for given S_1, the optimal values of x_1^* and r_1^* can be determined based on the procedures which will be discussed in the following section. In general at stage i, $i = 2, 3, \ldots$, n,

$$f_i(S_i) = \min_{Ri} \{x_i C_i(r_i) + f_{i-1}(S_{i-1}): 1-(1-r_i)^{xi} = R_i, \quad S_{i-1} = S_i R_i\} \quad (6)$$

where $S_n = 1$. As in stage 1, R_i^*, x_i^*, and r_i^* are determined for given S_i Then, the overall the optimal solution is found by tracking the optimal path form stage n to 1. In implementing the above procedure one needs to consider a set of grid points for the state and decision variables, together with some interpolation.

Note that for a selected R_i at stage i, the following optimization problem needs to be solved (subscript i is dropped for simlicity).

Problem II :

Minimize	$W(x, r) = xC(r)$	(7)
subject to	$1-(1-r)^x = R$	(8)
	$x \geqq 1$, integers	(9)

In the following section a soultion method for problem II will be discusssed.

4. Single-Stage Optimization

If we relax the integer restriction on x, then from Eqs. (8) and (9),

$$x = (ln\ Q)/ln(1-r) \quad (10)$$

where $0 < r \leqq 1-Q$, and $Q = 1-R$. Therefore, without integer restfiction, Problem II is equivalent to

Problem II-A :

Minimize	$Z(r) = ln\ Q \dfrac{C(r)}{ln(1-r)}$	(11)
subject to	$0 < r \leqq 1-Q$	

Assume that $Z(r)$ has a finite number of local minima. Although it is not difficult to find a function which violates this restriction, for most practical problems this presents no difficulty. Then, Problem II can be solved according to the following steps.

1. Solve problem II-A obtaining all the local minima r^j, $j = 1, 2, \dots, m$.
2. For each r^j, calculate x^j from Eq. (10).
3. Find $x^{j1} = [x^j]$ and $x^{j2} = x^{j1} + 1$ where $[x]$ is the largest integer less than or equal to x. If x^j is an integer, $x^{j1} = x^{j2} = x^j$.
4. Recalculate r as

$$r^{jk} = 1 - Q^{1/x^{jk}} \; ; \; j = 1, 2, \dots, m, \; k = 1, 2. \tag{12}$$

5. Choose x^{jk} and r^{jk} for which W in Eq. (7) is the minimum.

Essential to the success of the above procedure is the capability of identifying all the local minima of $Z(r)$. It is the author's experience that in many cases $Z(r)$ has only a few or less local minima so that graphical analyses may be used. In fact, for the cost functions in Table 1 the number of local minima is at most one. For a more systematic ways of implementing step (1), refer to [4] and the references cited therein. In the following, the above procedures are described in detail with an example.

The objective function $Z(r)$ does not necessarily possess such nice property as convexity or concavity. However, $Z(r)$ can be characterized based on the property of $Z'(r)$. That is,

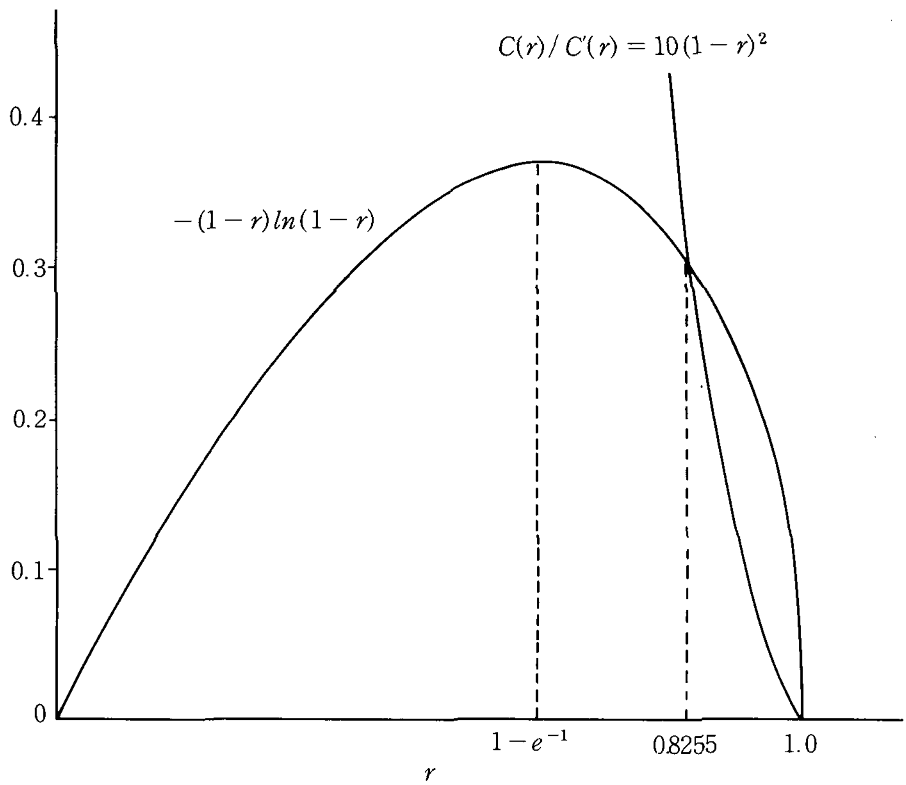

Fig. 2: $C(r)/C'(r)$ and $-(1-r)ln(1-r)$ for the Example.

$$Z'(r) = h(r)g(r) \tag{13}$$

where

$$h(r) = (ln\ Q)\ C'(r) / \{ (1-r)\ [ln\ (1-r)]^2 \},\ \text{and} \tag{14}$$

$$g(r) = (1-r) ln (1-r) + C(r) / C'(r). \tag{15}$$

Note that $h(r) < 0$ for $0 < r \leqq 1-Q$. Therefore, the sign and zeroes of $Z'(r)$ are determined by those of $g(r)$. The function $(1-r) ln (1-r)$ is convex and has a unique minimum at $r = 1-e^{-1}$. In many cases $C(r)/C'(r)$ has a rather simple functional form, sometimes even simpler than $C(r)$ itseif (e.g., see Table 1). The behavior of $g(r)$ is then easily identified by comparing $(1-r) \cdot ln(1-r)$ and $C(r)/C'(r)$. For instance, Figure 2 shows the graphs for $-(1-r) ln (1-r)$ and $C(r)/C'(r)$ when $C(r)$ is from [5] with $a = 0.1$ (see Table 1), and $1-Q = 0.98$. The two graphs meet at $r \cong 0.8255$.

From Figure 2 and Eqs. (13)−(15), the sign of $Z'(r)$ is negative for $0 < r < 0.8255$ and positive for $0.8255 \leqq r < 0.98$. In other words, $Z(r)$ is strictly decreasing up to $r = 0.8255$ and strictly increasing thereafter. We therefore conclude that $r = 0.8255$ is the global optimal solution to Problem II−A. The corresponding x is 2.24 from Eq. (10). Since this is not an integer, we compare the two cases, $x = 2$ and 3. When $x = 2$, the optimal value of r is 0.8568 from Eq. (12) and the total cost is 4.0567k. Similarly, when $x = 3$, $r = 0.7286$ and the total cost is 4.3365k. Therefore, the optimal solution to Problem II is $x^* = 2$ and $r^* = 0.8586$.

In general, $g(r)$ has at most few zeroes unless $C(r)/C'(r)$ is highly irregular. If $g(r)$ does not have any zeroes, then it is either positive or negative over the region $0 < r \leqq 1-Q$. The former implies that $Z'(r)$ is negative (or $Z(r)$ is strictly decreasing), and therefore, $r = 1-Q$ and $x = 1$ are optimal. In the latter case, however, the optimal value of x approaches infinity, which may be an indication of unrealistic assumptions on the cost function.

5. Concluding Remarks

The reliability and redundancy optimization is basically a mixed integer nonlinear programming problem. A simple approach is presented with an emphasis on solving the single-stage sub-problems. A fruitful area of future research may include an investigation of the behavior of the objective function (11), given the minimal set of assumptions on $C(r)$.

References

1. Aggarwal, K.K. and Gupta, J.S.,"On Optimizing the Cost of Reliable Systems," *IEEE Trans. Reliability,* Vol. 24, 205, 1975.
2. Gopal, K., Aggarwal, K.K., and Gupta, J.S.,"A New Method for Solving Reliability Optimization Problem," *IEEE Trans. Reliability,* Vol. 29, 36-37, 1980.
3. Kim, J.Y. and Frair, L.C.,"Optimal reliability Design for Complex Systems," *IEEE Trans. Reliability,* Vol. 30, 300-302, 1981.
4. McCormick, G.P.,"Finding the Global Minimum of a Function of One Variable Using the Method of Constant Signed Higher Order Derivatives," Nonlinear Programming 4, Academic Press, 223-243, 1981.

5. Misra, K.B. and Ljubojevic, M.D.,"Optimal Reliability Design of a System: A New Look," *IEEE Trans. Reliability,* Vol. 22, 255-258, 1973.
6. Nakagawa, Y. and Nakashima, K.,"A Heuristic Method for Determining Optimal Reliability Allocation," *IEEE Trans. Reliability,* Vol. 26, 156-161, 1977.
7. Sasaki, R., Okada, T., and Shingai, S., A New Technique to Optimize System Reliability, *IEEE Trans. Reliability,* Vol. 32, 175-182, 1983.
8. Tillman, F.A., Hwang, C., Fan, L., and Lai, K.C., Optimal Reliability of a Complex System, IEEE Trans. Reliability, Vol. 19, 95-100, 1970.
9 Tillman, F. A., Hwang, C, and Kuo,W.,"Determining Component Reliability and Redundancy for Optimumum System Reliability," *IEEE Trans. Reliaility,* Vol. 26, 162-165, 1977.
10. Tillman, F. A., Hwang, C., and Kuo, W., Optimization of Systems Reliability, Marcel Dekker. 1980.