Journal of the Korean
OR/MS Society
Vol. 10, No. 2, Oct. 1985

— Research Paper —

Sufficient Conditions for Zero Duality Gap of Lagrangean Relaxation

Sehun Kim*

Abstract

This paper presents sufficient conditions for zero duality gap of Lagrangean relaxation in mixed integer programming problems and discusses about an algorithm which updates multipliers using the dual varibles of the linear programming constructed by fixing integer variables.

1. Introduction

The Lagrangean relaxation method [2] has been quite successful in solving mixed integer programming problems. The Lagrangean multipliers usually have been updated using the subgradient optimization method [3].

It is known that if an optimal solution of a relaxed problem is a feasible solution of the original problem then it is an optimal solution of the original problem and there is no duality gap (when the relaxed constraints are equality constraints.) However it is quite hard to expect that an optimal solution of a relaxed problem satisfies the equality constraints of the original problem.

In this paper we present a stronger result which provides a sufficient condition for zero duality gap and provides an optimal solution in this case.

2. Duality gap

Consider the following mixed integer programming problem:

$$
\begin{aligned}
(P1)\quad t = \min\ & ax + cy \\
\text{s. t. }\ & Ax + By = b, && (2.1) \\
& Cx + Dy = d, && (2.2) \\
& y \geq 0,\ x \in I^m,
\end{aligned}
$$

where $A \in R^{p\times m}$, $B \in R^{p\times n}$, $C \in R^{q\times m}$, $D \in R^{q\times n}$, $a \in R^m$, $c \in R^n$, $b \in R^p$, $d \in R^q$, $y \in R^n$, and $I^m = \{ x \in R^m \mid x_i \text{ are integers} \}$.

Let us consider the following Lagrangean relaxation of $(P1)$:

*Dept. of Management Science, KAIST

$(P2)$ $L(u) = \min\ ax + cy + u(d - Cx - Dy)$

$$\text{s. t. } Ax + By = b,$$
$$y \geq 0 \text{ and } x \in I^m.$$

Then $\delta = t - \max_u L(u) \geq 0$ is the duality gap of this Lagrangean relaxation.

Define a point-to-set map $R^q \rightarrow I^m$ as follows:

$\alpha(u) = \{ x \in I^m \mid (x, y)$ for some y is an optimal solution of $(P2)$ for given $u \}$.

Then $\alpha(u)$ is upper semi-continuous (For a proof see [5, pp. 402]). For a given x, consider

$(P3)$ $\min\ cy$

$$\text{s. t. } By = b - Ax,$$
$$Dy = d - Cx,$$
$$y \geq 0.$$

Define a point-to-set map $I^m \rightarrow R^q$ as follows :

$\beta(x) = \{ u \in R^q \mid (w, u)$ for some $w \in R^p$ is a dual optimal solution of $(P3) \}$.

Then β is also upper semi-continuous. Consider a composite map $R^q \rightarrow R^q$ as follows: $\phi(u) = \beta(\alpha(u))$.

Theorem 1. If ϕ has a fixed point u^*, then there is no duality gap and (x^*, y^*) is an optimal solution of $(P1)$, where

(1) $x^* \in \alpha(u^*)$,

(2) y^* is an optimal solution of $(P3)$ for given x^* and

(3) there exists w^* such that (w^*, u^*) is a dual optimal solution of $(P3)$.

Poof. Since u^* is a fixed point of ϕ, there eixsts $(x^*, \underline{y}, y^*, w^*)$ such that $(x^*, \underline{y})$ is an optimal solution of $(P2)$ for given u^*, and y^* and (w^*, u^*) are, respectively, a primal optimal solution and a dual optimal solution of $(P3)$ for given x^*. Since (x^*, y^*) is a feasible solution of $(P1)$,

$$0 \leq \delta \leq ax^* + cy - L(u^*) = ax^* + cy^* - (ax^* + c\underline{y} + u^*(d - Cx^* - D\underline{y}))$$
$$= c(y^* - \underline{y}) - u^*(d - Cx^* - Dy^* + Dy^* - D\underline{y})$$
$$= (c - u^*D)(y^* - \underline{y}) \text{ since } (x^*, y^*) \text{ satisfies (2. 2)}$$
$$= (c - u^*D - w^*B)(y^* - \underline{y}) \text{ since } y^* \text{ and } \underline{y} \text{ satisfy (2. 1) for given } x^*$$
$$= -(c - u^*D - w^*B)\underline{y} \text{ from the complementarity of } (P3)$$
$$\leq 0 \text{ from the nonnegativity of } (c - u^*D - w^*B) \text{ and } \underline{y}.$$

Hence $\delta = 0$. Furthermore the above result shows that $ax^* + cy^* = (u^*)$.
Therefore (x^*, y^*) is an optimal solution of $(P1)$.

Define a composite map $I^m \rightarrow I^m$ as follows : $\psi(x) = \alpha(\beta(x))$.

Corollary 1. If ψ has a fixed point x^* then there is no duality gap and (x^*, y^*) is an optimal solution of $(P1)$ where y^* is an optimal solution of $(P3)$ for given x^*.

Proof. Let x^* be a fixed point of ψ. Then there exists (w^*, u^*) such that (w^*, u^*) is a dual optimal solution of $(P3)$ for given x^* and $(x^*, \underline{y})$ for some $\underline{y}$ is an optimal solution of $(P2)$ for given u^*. Hence u^* is a fixed point of ϕ. This completes the proof.

Since α and β are upper semi-continuous, ϕ and ψ are also upper semi-continuous (For a proof see [1, pp. 491]). Let $S = \{u \mid wB + uD \leq c$ for some $w\}$, the set of feasible dual variables of ($P3$).

Theorem 2. Suppose S and the feasible set of ($P2$) are bounded and $\phi(u)$ is convex for all u in S. Then there exists a fixed point of ϕ and there is no duality gap.

Proof. For any u, $\phi(u)$ is a subset of S. Since S is bounded ($P3$) is feasible for any x. Hence ($P1$), ($P2$) and ($P3$) are feasible problems. Since (P2) has a bounded feasible set, ($P1$), ($P2$) and ($P3$) have a finite optimal solution for any u in S. Therefore $\phi(u)$ is nonempty for any $u \in S$. Consider the point-to-set map $\phi : S \rightarrow S$. From the Kakutani's fixed point theorem [4] the existence of a fixed point is guaranteed when $\phi(u)$ is covex for all $u \in S$.

Among assumptions in Theorem 2, the boundedness assumption is not very restrictive in a practical problem since we can make S bounded by introductng artificial continuous variables with high costs into problem ($P1$) and we can make the feasible set of ($P2$) bounded by introducing artificial constraints. But it is hard to expect that the convexity assumption of $\phi(u)$ holds for all u. Rather we can test the existence of a fixed point using the algorithm described in Section 3. The following corollary partially restates the well-known results for linear programming.

Corollary 2. If x has no integerness condition and the boundedness assumptions in Theorem 2 hold then there is a fixed point of ϕ and there is no duality gap.

Proof. If x has no integerness condition then $\phi(u)$ is convex for all u.

3. Algorithm

Suppose that we have introduced artificial variables with high costs and artificial constraints into problem ($P1$) so that the set S end the feasible set of ($P2$) are bounded. Then $\psi(x)$ is nonempty for any $x \in I^m$. Since the fasible set of ($P2$) is bounded we can construct a bounded subset of I^m, T, such that the point-to-set map $\psi : T \rightarrow T$ is well defined. Consider the following algorithm : Choose a x^0 in T and generate a sequence $\{x^k\}$ such that $x^{k+1} \in \psi(x^k)$ for $k \geq 0$. Unlike the subgradient algorithm this algorithm updates the Lagrangean multipliers using the dual variables of ($P3$). Since T is bounded T is a finite set. Therefore the sequence will generate a cycle. If the period of the cycle is one then there exists a k such that $x^k \in \psi(x^k)$ and x^k is a fixed point of ψ.

Suppose that the period of the cycle r is greater than 1. Without loss of generality, suppose that $\{x^1, \ldots, x^r\}$ is a set of all different elements and $x^1 = x^{r+1}$. Let y^k be an optimal solution of ($P3$) for given x^k and u^k be the multiplier generated from ($P3$) for given x^k. Then $LB = \max L(u^k)$ is a lower bound.

If y^k containes no artificial variables then the solution (x^k, y^k) is a feasible solution. If there is no such k, then wecould not find a feasible solution. Otherwise find k with smallest objective function. Then (x^k, y^k) is a suboptimal solution with difference from the optimal solution less than or equal to $(ax^k + cy^k - LB)$.

4. Discussions

Suppose J^1 and J^2 are two subsets of the set of all constraints in problem (P1). Let δ^1 and δ^2 be the duality gaps of the Lagrangean relaxations constructed by relaxing constraints in J^1 and J^2, respectively. We can easily show that $\delta^1 \geqq \delta^2$ if $J^1 \supseteqq J^2$. Hence if the Lagrangean relaxation constructed by relaxing constraints in some J^1 has zero duality gap then the Lagrangean relaxation constructed by relaxing constraints in any subset of J^1 also has zero duality gap. It is an interesting question how to find a maximal set J^1 with zero duality gap. The results in Section 2 may be able to provide some answers for this question.

Suppose that our algorithm is combined with the branch-and-bound method to find an exact optimal solution. If our algorithm is applied to the partial problem at a node of the branching tree and finds a fixed point then it provides an exact optimal solution of the partial problem. Then this node can be fathomed. The subgradient algorithm does not have this property. In this sense our algorithm is expected to be more powerful than the subgradient algorithm when they are combined with the branch-and-bound method to find an exact optimal solution of a mixed integer programming.

References

1. M. Avriel, *Nonlinear Programming; Analysis and Methods,* Prentice-Hall, Englewood Cliffs, New Jersey (1976).
2. A.M. Geoffrion, "Lagrangean Relaxation for Integer Programming", *Mathematical Programming Study,* 2, 82-114, (1974).
3. M. Held, P. Wolfe and H. P. Crowder, "Validation of Subgradient Optimization", *Mathematical Programming,* 6, 62-88, (1974).
4. S. Kakutani, "A Generalization of Brouwer's Fixed Point Theorem", *Duke Mathematical Journal,* 8, 457-459, (1941).
5. W. I. Zangwill and C. B. Garcia, *Pathways to Solutions, Fixed Points and Equilibria,* Prentice-Hall, Englewood Cliffs, New Jersey (1981).