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— Research Paper —

A Multi-Product Multi-Facility Production Planning Model with Capacity Constraints

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Abstract

A multi-product multi-facility production planning model is studied in which known demands must be satisfied. The model considers concave production costs and piecewise concave inventory costs in the introduction of production capacity constraints. Backlogging of unsatisfied demand is permitted. The structure of optimal production schedules is characterized and then used to solve an illustrative numerical problem.

1. Introduction

Zangwill [6] has considered a multi-product, multi-facility production planning model, which is a linking together of certain single-facility models to form an acyclic network of various facilities. In the model, each facility except the first facility was allowed to receive inputs from raw materials and one or more facilities, then in each period manufacture a specific product on its own production line. The product was then stored in inventory until needed either to satisfy demands for the product or to supply input to other facilities. Assuming that demands for each product were known, he sought to determine the general form of the minimum cost production schedule that specified how much each facility in the network should produce. He further assumed that the production cost functions were concave and dependent upon the production in several different facilities, that the inventory cost functions were piecewise concave, and the backlog was permitted.

Lambrecht and Vander Eecken [4] have analyzed a facilities-in-series production planning model with capacity constraints on the last facility and involving concave cost functions. He assumed that one unit of production at any facility required as input one unit of production from the preceding facility and production was instantaneous, and that no backlog was permitted.

Numerous studies for single-product single-facility production planning problems with capacity constraints and with (or without) backlog have been done by a number of authors (see, for example, Baker *et al* [1] Florian and Klein [2] Florian *et ol* [3] and Love [5]).

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In this paper, the model treated by Zangwill [6] is considered in the introduction of capacity constraints for production at each facility. The objective is to exploit the structure of optimal schedules that specify how much each facility in the network should produce so that the total cost is minimized. As a useful description of the structure of optimal schedules, the dominant set defined as the set of all extreme points will be proved, based on the partitioning of the feasible solutions set into basic sets done by Zangwill [6], to contain an optimal schedule. Each component of such an optimal schedule is associated with a specific facility and consists only of subplans each representing "a capacity constrained sequence" defined in Florian and Klein [2]. Thereupon, an optimal schedule out of the dominant set can be found by doing recursively a shortest path search for a specific component associated with each facility, starting with the search for the last facility's schedule to satisfy the given demands. A numerical example is to be presented.

2. Model Formulation

Consider a M-product M-facility production planning problem with a planning horizon N. The individual facilities are linked together to form an acyclic network depicted in Fig. 1, where $R^j = (R_1^j, R_2^j, \dots, R_N^j)$, $R_i^j \geqq 0$, represents the market requirements (demand) vector for facility j ($j = 1, 2, \dots, M$) over the planning horizon N. It is assumed that all demands R_i^j are fixed and known.

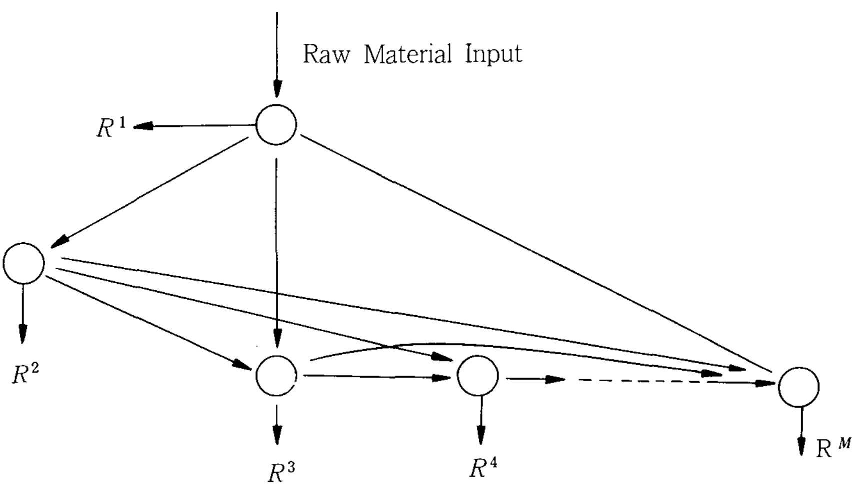

Fig. 1: The Acyclic Network with M Facilities.

Each facility can receive inputs from either raw materials or lower numbered facilities. It cannot receive inputs from itself or higher numbered facilities. However, each facility can supply only higher numbered facilities or demands for its own product. Facility 1 receives raw materials only and facility N supplies demands for its own product only.

Let X_i^j, $X_i^j \geqq 0$, be the ptoduction completed in period i at facility j and I_i^j the inventory at the end of period i in facility j. Let a_{jk}, $\geqq 0$, be the number of units of facility j's product required to produce one unit of facility h's product. It is assumed that there is no time loss in transmission of goods from one facility to another, while each facility j can have a time lag λ_j in production and thereby production started in period i is completed in period $i+\lambda_j$ Then, the amount desired out of facility j in period i as inputs to other facilities is $\sum_{h=j+1}^{M} a^{jh} X^h_{i+\lambda_h}$ and the total demand on facility j in period i and the inventory level, denoted respectively by Y^j_i and I^j_i, are

$$Y^j_i = R^j_i + \sum_{h=j+1}^{M} a^{jh} X^h_{i+\lambda_h} \quad \text{and}$$

$$I^j_i = \sum_{h=1}^{i} (X^j_i - Y^j_h) \quad \text{for all } i \text{ and } j.$$

It is also assumed that each facility can backlog total demand for its product a certain fixed integral number of periods. Let α_j represent the number of periods of backlog permitted for facility j. Then, the backlog limit for facility j is

$$I^j_i \geqq - \sum_{h=i-\alpha_j+1}^{i} Y^j_h .$$

As noted in Zangwill [6], it is always possible to append an artificial extra no cost period and assume $I^j_N = 0$ for all j. (We assume $I^j_0 = 0$ for all j). Finally, assume capacity restrictions on each period's production at eact facility, so that $0 \leqq X^j_i \leqq C^j_i$ and further $\sum_{h=1}^{i-\alpha_j} Y^j_h \leqq \sum_{h=1}^{i} C^j_h$, for all i and j, where C^j_i denotes the capacity restriction on the production in period i at facility j.

Let $Z = (X_1, X_2, \dots, X_M) = (X^1_1, X^1_2, \dots, X^1_N, X^2_1, X^2_2, \dots, X^2_N, \dots, X^M_1, X^M_2, \dots, X^M_N)$ denote the production schedule for the entire network, where $X^j = (X^j_1, X^j_2, \dots, X^j_N)$ represents the production schedule (vector) for facility j. The problem is now to find a production schedule Z, called optimal, which minimizes the piecewise concave function

$$F(Z) = P(Z) + \sum_{j=1}^{M} \sum_{i=1}^{N} H^j_i(Z)$$

subject to

$$I^j_i = \sum_{h=1}^{i} (X^j_h - Y^j_h)$$

$$Y^j_i = R^j_i + \sum_{h=j+1}^{M} a^{jh} X^h_{i+\lambda_h}$$

$$I^j_i \geqq - \sum_{h=i-\alpha_j+1}^{i} Y^j_h \qquad \text{(A)}$$

$$0 \leqq X^j_i \leqq C^j_i$$

$$\sum_{h=1}^{1-\alpha_j} Y^j_h \leqq \sum_{h=1}^{i} C^j_h$$

$$I^j_0 = 0 = I^j_N, \quad \text{for all } i \text{ and } j,$$

where $P(Z)$ represents the concave production costs among M facilities, and $H_i^j(Z) = H_i^j(I_i^j(Z)) = H_i^j(I_i^j)$ is the concave inventory cost on interval $(-\infty, 0]$ and on the interval $[0, +\infty)$ but need not be concave on the interval $(-\infty, +\infty)$. Note that since I_i^j is a linear function of Z, the relationship is expressed as $I_i^j = I_i^j(Z)$.

Let X be the set of all the associated feasible production vectors. Then, X is a bounded polyhedral set and is thereby convex and compact. Any Z in X is called feasible.

3. Construction of Dominant Set

Zangwill [6] has shown that for a general acyclic network composed of facilities with no capacity constraints there exists an optimal schedule in a set called the dominant set, which is the set of extreme points out of the compact convex sets on which $F(\cdot)$ is defined cancave. This was verified based on the property that the set of all feasible production schedules can be partitioned into disjoint subsets, called "basic sets", in which the i^{th} components $I_i^j(Z)$ of each $I^j(Z)$ (each Z) in a basic set all have the same sign for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. This partitioning will also play a central role in our approach.

Following Zangwill [6], such a basic set B_k is defined as

$$B_k = \{Z \mid Z \in X \,;\, (-1)^{k_i^j}\ I_i^j(Z) \geqq 0 \text{ for all } i \text{ and } j\},$$

where $k = (k_i^j)$ is a $M \cdot N$ component vector with $M(j-1) + i^{th}$ component k_i^j such that k_i^j is $+1$ or 0 only, and $k \in K = \{k = (k_i^j) \mid k_i^j = +1 \text{ or } 0 \text{ only}\}$. Then, B_k is the set of all feasible schedules Z that in facility j period i give rise to nonnegative inventories if $k_i^j = 0$ or nonpositive inventories if $k_i^j = +1$. Evidently, each B_k is compact and convex, and hence $F(Z)$ being concave on a particular B_k can be minimized on B_k at an extreme point of B_k. Furthermore, X is the union of all $2^{M \cdot N}$ basic sets ; that is, $X = \bigcup_{k \in K} B_k$. Letting $E[B_k]$ donote the extreme points of a basic set B_k, $F(Z)$ must be minimized on X at a point in $D = \bigcup_{k \in K} E[B_k]$, where D is called the "dominant set".

For convenience, let $Z^j = (X^j, X^{j+1}, \ldots, X^M)$ be defined as a partial production vector in facilities j through M. The vectors $Y^j = (Y_1^j, Y_2^j, \ldots, Y_N^j)$ and $R^j = (R_1^j, R_2^j, \ldots, R_N^j)$ represent respectively the total demand and the market requirements for facility j. Then, a partial production vector Z^h is said to be feasible if the constraints of the problem (A) hold for all $j \geqq h$ and all i. Similarly, if Z^{h+1} is feasible, X^h is said to feasibly supply Z^{h+1} if the partial production vector $Z^h = (X^h, Z^{h+1})$ is feasible.

For constructing D, our approach is now started with Lemma 1 (referring to Zangwill [6]).

Lemma 1.

If two vectors Z and $\bar{Z}$ are in the same basic set B_k, then $I_i^j \geqq 0$ iff $\bar{I}_i^j \geqq 0$ for all i and j. and conversely.

Let $D[r]$ be the dominant set if a single-facility model has market requirements $r = (r_1, r_2, \ldots, r_N)$. As done in Zangwill [6], by using $D[r]$ recursively the dominant set for the entire M facilities acyclic network can then be constructed. Consider partial dominant set D^b constructed from facilities b through M. D^b, $b \geq 1$, can then be constructed by induction when D^{b+1} has been constructed.

Let Z^{b+1} be in D^{b+1} . Z^{b+1} does completely specify Y^b. Letting $Y^b(Z^{b+1})$ donote Y^b' s dependence on Z^{b+1}, $D[Y^b(Z)^{b+1})]$ would be the set of all X^b in the dominant set if facility b were considered to be a single facility facing market requirements of $Y^b(Z^{b+1})$. Therefore, for each $Z^{b+1} \varepsilon D^{b+1}$, $D[Y^b(Z^{b+1})]$ can be constructed. Therewith, D^b consists of all partial production vectors (X^b, Z^{b+1}) such that X^b is in $D[Y^b(Z^{b+1})]$ for a $Z^{b+1} \varepsilon D^{b+1}$. Thus, $D^b = \bigcup_{Z^{b+1} \varepsilon D^{b+1}} \{(X^b, Z^{b+1}) \mid X^b \varepsilon D[Y^{b+1})]\}$. This construction is continued until D^1 is constructed.

Before demonstrating that D^1 is the dominant set, it is necessary to refer to the set of all feasible schedules consisting only of "capacitry-constrained sequences", defined in Florian and Klein [2], which characterizes all the extreme points of the solution set for a single-product single-facility model with capacity restrictions. In a capacity constrained sequence, the production level in at most one period d, $u+1 \leq d \leq v$, is partial production (*i. e.*, $0 < X_d^b < C_d^b$ in a feasible schedule $\bar{X}^b$ for facility b) and all other production levels are either zero or at their capacities, and further $I_t^b \neq 0$ for $u+1 \leq t < v$ but $I_u^b = 0 = I_v^b$ for $0 \leq u < v \leq N$. The associated solution set characterization is specified in Theorem 1.

Theorem 1.

(a) Given a facility b, $1 \leq b \leq M$, if $I_h^b = 0$ for some $h \ \varepsilon \ \{1, 2, \ldots, N\text{-}1\}$ and $\sum_{j=h+1}^{i} C_j^b \geq \sum_{j=h+1}^{i-\alpha+1} Y_j^b$ $(i = h + \alpha_b, \ldots, N-1)$, then the original problem is decomposed into two parts; one part for the first h periods and the other for the last $(N-h)$ periods.

(b) $\bar{X}^b$ and $\hat{X}^b$ are distinct feasible schedules and $X^b = \frac{1}{2}(\bar{X}^b + \hat{X}^b)$, then $\bar{X}^b$ and $\hat{X}^b$ share all the regeneration points (zero inventory points) of X^b.

(c) A feasible solution is an extreme point iff it is composed solely of capacity constrained sequences.

The results of Theorm 1 will be useful in proving that D^1 is the dominant set D, since the solution set characterization for a single facility model can be extended by induction to the general M facilities network. In fact, given a partial production vector Z^{b+1} and the total demand on facility b, $Y^b = Y^b(Z^{b+1})$, then a feasible schedule X^b consists only of capacity constrained sequences iff it is in $D[Y^b(Z^{b+1})]$. In other words, letting Z be in D^1, and X^b and Z^{b+1} be components of Z so that X^b is in $D[Y^b(Z^{b+1})]$, then X^b consists solely of capacity constrained sequences for the demand $Y^b(Z^{b+1})$.

Lemma 2.

Let $\Omega(Y^j)$ denote the set of all feasible production schedules for a demand vector $Y^j = (Y_1^j, Y_2^j, \ldots, Y_N^j)$ at a facility $j (j = 1, 2, \ldots, M)$. Then, $Y^j \ \varepsilon \ \Omega(Y^j)$ and further $Y^j \ \varepsilon \ D[Y^j]$.

Proof. For every production schedule $X^j \ \varepsilon \ \Omega(Y^j)$, it holds that $\sum_{h=1}^{t-\alpha_j} Y_h^j \leq \sum_{h=1}^{t} X_h^j$ and $\sum_{h=1}^{t-\alpha} Y_h^j \leq \sum_{h=1}^{t} C_h^j$ for all $t = 1, 2, \ldots,$, N. Therefore, Y^j is included in $\Omega(Y^j)$. Furthermore, since $\sum_{h=1}^{N} X_h^j = \sum_{h=1}^{N} Y_h^j$ from the constraint $I_N^j = 0$ and hence the set $\Omega(Y^j)$ is compact and convex, it is evident that the vector Y^j is an extreme point of the feasible set $\Omega(Y^j)$.

Lemma 2 indicates that in the problem (A) the total demand vector Y^b at facility b is itself an extreme point of the set of all feasible production schedules to satisfy the demand Y^b, and hence it consists only of capacity constrained sequences coresponding to the associated capacity constraints. This leads to Lemma 3.

Lemma 3.

Let Z^{b+1}, $\bar{Z}^{b+1}$, $\hat{Z}^{b+1}$ be feasible partial production vecors such that $\bar{Z}^{b+1} \neq \hat{Z}^{b+1}$, and $Z^{b+1} = \frac{1}{2}(\bar{Z}^{b+1} + \hat{Z}^{b+1})$, $b \geqq 1$. Assume X^b feasibly supplies Z^{b+1} and it consists only of capacity constrained sequences for $Y^b(Z^{b+1})$. Then, there exist production vectors $\bar{X}^b$ and $\hat{X}^b$ that feasibly supply $\bar{Z}^{b+1}$ and $\hat{Z}^{b+1}$, respectively, such that $(X^b, Z^{b+1}) = \frac{1}{2}[(\bar{X}^b, \bar{Z}^{b+1}) + (\hat{X}^b, \hat{Z}^{b+1})]$. Furthermore, $I_i^b = 0$ iff $\bar{I}_i^b \geqq 0$, and $I_i^b \geqq 0$ iff $\hat{I}_i^b \geqq 0$, for all $i = 1, 2, \ldots, N$.

Proof. Z^{b+1} is not an extreme point. Theorm 1 indicates thereby that there exists at least one facility j, $b+1 \leqq j \leqq M$, having a production sequence S_{uv}^j for $0 \leqq u < v \leqq N$ (definedas $S_{uv}^j = \{ X_i^j,\ i = u+1, \ldots, v \mid I_u^j = 0 = I_v^j\ ;\ I_i^j \neq 0$ for $u < i < v \}$ for a feasible schedule X^j), in which there are at least two periods t_1 and t_2, $u+1 \leqq t_1 < t_2 < v$, such that $0 < X_{t_1}^j < C_{t_1}^j$ and $0 < X_{t_2}^j < C_{t_2}^j$. According to Lemma 2, there is an extreme point X^b for $Y^b(Z^{b+1})$ such that $X^b = Y^b(Z^{b+1})$. It is then seen from the capacity constraints of the problem (A) that the two periods t_1 and t_2 are also partial production periods in the schedule X^b ; that is, $0 < X_{t_1}^b < C_{t_1}^b$ and $0 < X_{t_2}^b < C_{t_2}^b$

Let $\varepsilon = \frac{1}{2}\min\{X_{t_1}^b, C_{t_1}^b - X_{t_1}^b, X_{t_2}^b, C_{t_2}^b - X_{t_2}^b\}$. Define $\bar{Y}_{t_1}^b = Y_{t_1}^b - \varepsilon$, $\bar{Y}_{t_2}^b = Y_{t_2}^b + \varepsilon$, $\hat{Y}_{t_1}^b = Y_{t_1}^b + \varepsilon$, $\hat{Y}_{t_2}^b = Y_{t_2}^b - \varepsilon$, and $\bar{Y}_t^b = \hat{Y}_t^b = Y_t^b$ for all t except at t_1 and t_2, where Y_t^b is the t^{th} component of $Y^b(Z^{b+1})$. Since $\varepsilon > 0$, we can have the associated feasible production vectors $\bar{X}^b$ and $\hat{X}^b$ in which $\bar{X}^b = X^b - \varepsilon U_{t_1} + \varepsilon U_{t_2}$ and $\hat{X}^b = X^b + \varepsilon U_{t_1} - \varepsilon U_{t_2}$, where U_j is a N component vector with a unity element in the i^{th} position and zeros elsewhere. It then follows that $X^b = \frac{1}{2}(\bar{X}^b + \hat{X}^b)$ and further $(X^b, Z^{b+1}) = \frac{1}{2}[(\bar{X}^b, \bar{Z}^{b+1}) + (\hat{X}^b, \hat{Z}^{b+1})]$.

It remains to prove that $I_i^b \geqq 0$ iff $\bar{I}_i^b \geqq 0$, and $I_i^b \geqq 0$ iff $\hat{I}_i^b \geqq 0$ for $i = 1, 2, \ldots, N$. If $I_i^b \geqq 0$, it implies that $\sum_{h=1}^{i} X_h^b \geqq \sum_{h=1}^{i} Y_h^b$. Since $\bar{I}_i^b = \sum_{h=1}^{i} (\bar{X}_h^b - \bar{Y}_h^b)$, $\bar{I}_{t_1}^b = \sum_{h=1}^{t_1-1} (X_h^b - Y_h^b) + \{(X_{t_1}^b - \varepsilon) - (Y_{t_1}^b - \varepsilon)\} \geqq 0$, and likewise, $\bar{I}_{t_2}^b \geqq 0$, and hence $\bar{I}_i^b \geqq 0$ for all i. $\hat{I}_i^b \geqq 0$ Similarly for all i. This completes the proof.

In the same manner, $I_i^b \leqq 0$ implies $\bar{I}_i^b \leqq 0$ and $\hat{I}_i^b \leqq 0$. We will now prove that D^1 is the dominant set D.

Theorem 2.

The set D^1 is the dominant set, *i. e.*, $D^1 = D$.

Proof. Applying Lemmas 1 and 3, it can be easily proved by following the proof steps of Theorem 1 in Zangwill [6].

Theorem 2 describes how to determine an optimal solution. However, it seems difficult to obtain an efficient algorithm to find a solution vector in the given M facilities case. In particular, there is a major difficulty in generating the set of all the extreme points at each facility, and moreover, the problem complexity is greatly depenent upon the associated network structure. In fact, Florian *et al* [3] have shown that even the capacity-contsrained single-facility case is in the class of NP-complete problems. Therefore, we will give a simple numerical example to illustrate only how to determine an optimal solution.

4. Numerical Example

Consider a 3-facility 3-period problem depicted in Fig. 2.

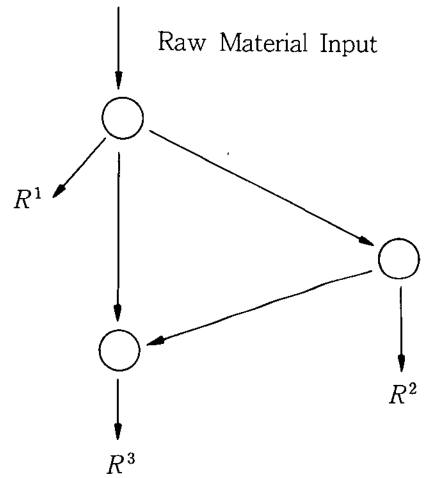

Fig. 2 : The Acyclic Network with 3-Facilities.

Let facilities 1, 2, and 3 have the respective market requirements $R^1 = (2, 3, 4,), R^2 = (1, 2, 3)$ and $R^3 = (3, 4, 5)$, and the respective capacity restrictions of $C_i^1 = 14$, $C_i^2 = 8$ and $C_i^3 = 7$ for all $i = 1, 2, 3$. Assume, for convenience, that no backlog is permitted, that $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and that $a^{jh} = 1$ for all $j = 1, 2$. Then, rather than the general shortest path algorithm (illustrated with a special case by Florian and Klein [2]), the tree-search algorithm of Baker *et al* [1] (which was shown more practical for reasonable sized problems without backlog) shall be applied for this problem. In fact, the algorithm was developed based on the optimal solution properties for capacity-constrained single-facility problems without backlog that if $(X_1^b, \ldots, X_N^b)$ represents an optimal production schedule at facility b, then $I_{i-1}^b\ (C_i^b - X_i^b)\ X_i^b = 0$ for every $i (i = 1, 2, \ldots, N)$ and further $X_t^b = min\ \{C_t^b, \sum_{i=t}^{N} R_i^b\}$, where $t = max\ \{i | X_i^b > 0\}$ and R_i^b represents the demand at facility b in period i.

Let the production and inventory cost functions be given as

$$P_i^1(X) = p(X), \qquad H_i^1(I) = 3\ H(I),$$
$$P_i^2(X) = 2\ p(X), \qquad H_i^2(I) = 2\ H(I),$$
$$P_i^3(X) = 3\ p(X), \text{ and } \qquad H_i^3(I) = H(I), \text{ for all } i,$$

where P_i^j is the production cost function of facility j in period i, $p(X) = 3\delta(X) + 5X$ for $X \geqq 0$, $H(I) = 3I$ for $I \geqq 0$, and $\delta(X)$ is the index function for production set-up with value 1 for positive X and zero elsewhere.

The partial dominant set D^3 is $D^3 = \{(3, 4, 5), (7, 0, 5), (5, 7, 0), (7, 5, 0)\}$. The associated D^2 consists of vectors that are written in the form $Z^2 = (X_1^2, X_2^2, X_3^2/X_1^3, X_2^3, X_3^3)$; that is,

$$\begin{aligned} D^2 = \{&(4,6,8/3,4,5),\ (8,2,8/3,4,5),\ (4,8,6/3,4,5),\ (8,8,2/3,4,5),\\ &(8,2,8/7,0,5),\ (8,8,2/7,0,5),\ (8,8,2/5,7,0),\ (8,7,3/5,7,0),\\ &(7,8,3/5,7,0),\ (8,7,3/7,5,0),\ (7,8,3/7,5,0),\ (8,8,2/7,5,0)\}. \end{aligned}$$

Likewise, $D^1 = D = \{(11, 14, 14/4, 6, 8/3, 4, 5), (14, 11, 14/4, 6, 8/3, 4, 5), (14, 14, 11/4, 6, 8/3, 4, 5), (14, 11, 14/8, 2, 8/3, 4, 5), (14, 14, 11/8, 2, 8/3, 4, 5), (11, 14, 14/4, 8, 6/3, 4, 5), (14, 11, 14/4, 8, 6/3, 4, 5), (14, 14, 11/4, 8, 6/3, 4, 5), (14, 14, 11/8, 8, 2/3, 4, 5)\}$.

Since one of the 9 schedules in D is optimal, the cost of each one can be evaluated to find the least expensive schedule, which is optimal. Therefore, in this problem extreme point $(14, 14, 11/8, 8, 2/3, 4, 5)$ is the optimal solution, which gives the total cost of 618. However, the identification of the dominant set D is, in general, a tedious combinatorial problem.

Note that given a demand vector, the algorithm of Baker *et al* [1] can be applied to find an optimal solution, but may not be possible to search for some other alternatives. For example, given $X^3 = (3, 4, 5)$ as the demand for facility 2, the algorithm found the optimal solution, $X^2 = (4, 6, 8)$, and then selected $X^1 = (11, 14, 14)$ as the optimal solution for the demands X^2 and X^3 at the facility 1. However, the schedule $Z = (X^1, X^2, X^3)$ is not optimal. This is because the algorithm could not select the alternative extreme point $\bar{X}^2 = (8, 8, 2)$ for the demand $X^3 = (3, 4, 5)$. This implies that in general, the algorithm may not be practical for the general acyclic network of M facility case.

5. Conclusion

In this paper we have exploited the structure of optimal production schedules in the general M facility case. Every optimal schedule consists only of the components each representing an extreme point of the set of all feasible production schedules for each associated demand vector on a facility in the given acyclic network. Such an extreme point is composed solely of capacity-constrained sequences. Thereupon, given a facility those extreme points for each of the associated total demand vectors can be searched by following the shortest path search mechanism shown in Florian and Klein [2]. However, the identification of all the extreme points is, in general, a tedious combinatorial problem, so that it seems difficult to obtain an efficient algorithm for an optimal schedule. Nevertheless, the exploited structure of optimal production schedules may have many practical meanings in solving small sized assembly networks.

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