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—Research Paper—

Market and Shadow Prices in a Pure Consumption Economy With Institutional Price Constraints

Sehun Kim*

Abstract

When an economy has institutional price constraints the ralationship between market prices and shadow prices is not yet fully investigated. A pure consumption economy is considered where market prices guide the consumption behavior and shadow prices measure the social value of resources. In this case we show that if the utility function is additively separable there exists a complementarity relation between the difference of the market price and the shadow price and the difference of the market price and the regulated upper(or lower) bound.

1. Introduction

Since 1950's optimization models have been widely used for economic planning which is a problem of a single-decision maker. Recently, it has become apparent that an optimization model is not well suited to situations where many agents independently maximize their profits or their own utility functions. In this case, an equilibrium model is required to describe the economy.

If there are perfect competitions in both factors and commodity market, then the supply responses will be the same whether we treat the production side of the economy as an optimization problem or as a decentralized market. In this case, the market equilibrium prices are identical to the shadow prices, *i.e.*, primal-dual equivalence holds. This implies that the market price of a resource should be equal to zero if there is an excess supply of that resource. Using the primal-dual equivalence, the equilibrium solution can be solved through a sequence of linear programmings [Manne, Chao and Wilson (1980), Kim (1981) and Chao, Kim and Manne (1982)].

One of the recently recognized major distortions in modern economies is that governments pursue public policies related to price regulations. Examples are minimum wage rates, ceiling prices for particular commodities, and linked price system between various commodities. A

* Dept. of Management Science, KAIST

serious difficulty arises when one attempts to implement such institutional price constraints into the model since the primal-dual equivalence breaks down.

The presence of such institutional price constraints implies that the market price and the shadow price of a commodity will not necessarily coincide. It allows us to explain the simultaneous existence of excess supply of a commodity and yet a positive market price. Since such institutional price constraints will typically distort the economy away from an optimal allocation of resources, it is an interesting subject to find the next best solution satisfying the institutional price constraints. Our aim in this paper is to present a mathematical economic model which can provide answers for these questions.

The presence of price regulations will in general cause market imbalance. To make the disequilibrium situation rather permanent, it is necessary to employ a mechnism to stabilize the market. Mathiesen (1980) considered this problem and formulated it as a complementarity model by imposing a counteracting mechanism which is a compensating program. It has a public sector which collects taxes and distributes them as a compensating program for unemployed resources. Imam and Whalley (1979) introduced a different counteracting mechanism which is the government intervention enforcing particular agents to purchase commodities through a government marketing agency.

Hansen and Manne (1977) considered the minimum and maximum price regulations. They assumed that there exists a set of shadow prices and that there are complementarity relations between the difference of the market price and the shadow price and the difference of the market price and the regulated upper (or lower) price bound. They did not discuss the validity of this assumption.

Kim (1984) considered a decentralized economy where market prices guide the decentralized behavior of each activity and the shadow prices measure the social values of resources. To measure the social values, he introduced a social objective criterion. Hence, his approach could be regarded as a central economic price control with institutional price constraints for an economy with decentralized producers. Each producer behaves as a price taker maximizing his profit and the planner maximizes the social objective criterion while observing each producer's behavior. He considered only the production side of the economy.

In our paper, we consider an economy with a single consumer group behaving as a price taker. Market prices guide the consumption behavior of the consumer group and the shadow prices measure the social values of resources. A social objective criterion is introduced into this economy to measure the social values of resources. Hence, there are two decision makers, the planner and the aggregated consumer group. The planner wants to choose a market price system satisfying the institutional price constraints to maximize the social objective criterion while observing the behavior of the consumer group and resource constraints. Hence this model could be regarded as a central economic price control with institutional price constraints for a pure consumption economy.

One of the most interesting results of our analysis is that we can prove that with the assumption of "want independence" [Frisch (1959)] which implies the additive separability of utility function the assumption made in Hansen and Manne (1977)…the existence of complementarity realtion between the difference of the market price and the shadow price and the difference of the market price and the regulated upper (or lower) bound…does actually hold in our model.

Furthermore, if the utility function is the logarithm of a Cobb-Douglas utility function, then this model can be converted to an equivalent linear program. The solution of this linear program can be found in an analytic form. However, if production side of the economy is introduced into our model, these results may not hold. Combining production side and consumption side is an interesting future research.

We shall begin by introducing the formulation of our model in section 2. In section 3, necessary optimality conditions are provided. Here, the concept of shadow prices of our model is introduced. Especially, the separable utility function case is discussed in section 4. Section 5 discusses the logarithmic utility function case. An example is used to illustrate graphically our model in section 6. Our results are discussed and future research works are suggested in the last section.

2. Formulation of the Model

Let us consider a pure consumption economy with n consumable goods. A vector b denotes the amount of goods available to the economy. We will assume that $b_i>0$ for all $i=1,\cdots n$. The consumption behavior of the economy is determined by a global utility function which is assumed to be differentiable, strictly increasing and concave. Also assume that $\partial U(z)/\partial z_i\rightarrow\infty$, as $z_i\rightarrow 0$. This guarantees that $z_i>0$ for all i. Without price regulations, the economy can be modeled as the following optimization problem:

(Problem 1)

$$\begin{aligned} \text{Maximize} \quad & U(z), \\ \text{subject to} \quad & z\leq b, \\ & z\geq 0. \end{aligned}$$

The primal solution, z, and the shadow price, p, satisfy the following conditions:

$$b-z\geq 0, \tag{1}$$

$$\nabla U(z)=p, \tag{2}$$

$$p'(b-z)=0. \tag{3}$$

When there are no institutional price constraints, conditions (1)—(3) constitute a set of market equilibrium conditions with a market price system, p. Conditions (1) is simply the material balancing constraint. Condition (2) describes the consumers' behavior. They consume a consumable good until their marginal utility with respect to that commdity becomes equal to the market price. Condition (3) describes the market behavior of prices. If there is an excess supply of a commodity then its market price should be equal to zero. This is the property that distinguishes a perfectly competitive market from a regulated market. Since Problem 1 and the system (1)—(3) are equivalent, without any insitutional price constraints the optimal planning is equivalent to a competitive market.

Let us suppose now that institutional price constraints of the following form are introduced to the economy:

$$Dp\geq 0, \tag{4}$$

where D is an m by n matrix, where m is the number of institutional price constraints. Since we are dealing with a general equilibrium problem, only the relative prices of goods are determined. Hence, all price constraints should be homogeneous like (4).

There are no restrictions on the type of the matrix D. Let commodity n have been chosen as a numeraire good. Then the minimum price regulation of a commodity i can be represented as $p_i - f_i p_n \geq 0$, where f_i is a constant. The ceiling price regulation is $-p_i + f_i p_n \geq 0$. If we use both the minimum and ceiling price regulations, we can represent a fixed price regulation. If a price is linked to others, then we could introduce the following two price constraints:

$$p - \sum_{j \neq i} f_j \, p_j \geq 0 \quad \text{and} \quad -p_i + \sum_{j \neq i} f_j \, p_j \geq 0.$$

If the system of shadow prices determined from Problem 1 satisfies the institutional price constraints (4), then this shadow price system is the market price system. Otherwise, we should find another set of market prices. In this case, the perfectly competitive equilibrium market condition (3) is distorted by the institutional price constraints and does not hold any further. The system of remaining conditions (1), (2) and (4) constitutes a feasible set for the consumption levels and the market prices. It is not clear which one is a solution among all possible values of (z,p) in this set. To define a solution, we need to impose additional mechanism into this set.

Unlike the market equilibrium models like Hansen and Manne (1977), Mathiesen (1980) and Imam and Whalley (1979), our model is a central economic price control model. The planner now wants to choose a system of market prices which maximizes a social objective criterion while observing the behavior of consumers. We employ the consumers' utility function as the social objective criterion. Then our model can be formulated as follows:

(Problem 2)

$$\begin{aligned} \text{Maximize} \quad & U(z), \\ \text{subject to} \quad & z \leq b, \\ & \nabla U(z) = p, \\ & Dp \geq 0, \\ & z \geq 0, \ p \geq 0. \end{aligned}$$

The market prices are also primal variables in this model but the objective function depends only on z.

3. Necessary Optimality Conditions

Since the second constraint of Problem 2 may not provide a convex feasible set, Problem 2 is not, in general, a convex program. Hence it is difficult to find a set of necessary and sufficient conditions for the global optimal solution. But we can find a set of necessary conditions through the Kuhn-Tucker theorem.

Theorem 1. Let (z,p) be an optimal solution of Problem 2 and satisfy the Kuhn-Tucker constraint qualification, then there exist nonnegative vectors y and w such that

$$y'(b-z) = 0, \tag{5}$$

$$p - y = -\nabla^2 U(z) D' w, \tag{6}$$

$$w'Dp=0. \tag{7}$$

Proof. From the Kuhn-Tucker theorem, there exist vectors y, v, and w such that

$$\nabla U(z)-y-\nabla^2 U(z)v=0, \tag{8}$$

$$v+D'w=0, \tag{9}$$

$$y'(b-z)=0, \tag{10}$$

$$w'(Dp)=0, \tag{11}$$

$$y\geq 0,\quad w\geq 0,$$

Where the fact that $z_i>0$ and $p_i=\partial U(z)/\partial z_i>0$ has been used to guarantee the equality in (8) and (9). By eliminating v from (8) and (9), we can derive the equation (6). QED

Since y_i is the dual variable associated with the material balancing constraint (1), we may call it the shadow price of the commodity i. Condition (5) says that the shadow price of a resource should be equal to zero if there is an excess supply of that resource. (Notice that there is no zero market price condition for a resource with excess supply.) Condition (6) shows the price wedge between the mar ket price and the shadow price. The wedge is proportional to the second derivative of the utility function.

4. Separable Utility Function Case

Let us suppose that the utility of consumers has the property of "want independence" [Frisch (1959)] which implies that the utility function is additively separable in its arguments.

$$U(z)=\sum_{i=1}^{n} u_i(z_i). \tag{12}$$

Then (6) reduces to

$$y_i=p_i-(p_i e_i/z_i)\sum_{j=1}^{m} w_j d_{ji}, \tag{13}$$

where d_{ij} is the (i,j)-th element of the matrix D and $e_i=-d\ \log(du_i(z_i)/dz_i)/d\ \log\ z_i$, the elasticity of the marginal utility with respect to the consumption level of commodity i. This is not, in general, a constant. Especially, if $U(z)$ is the addi-log utility function [Houthakker (1960)],

$$U_i(z_i)=a_i z_i^{c_i},\ \text{with}\ c_i<1,\ c_i\neq 0,\ a_i c_i>0,$$

then $e_i=1-c_i$ which is a constant. If $U(z)$ is the logarithmic utility function which is the logarithm of a Cobb-Douglas utility function, then $e_i=1$ for all i.

Now let us consider a special case where there is one minimum or ceiling price constraint for each commodity with respect to a numeraire good. Let the commodity n be the numeraire good. Without loss of generality, we assume that the first k commodities have minimum price constraints and the rest (except the numeraire good) have ceiling price constraints. This can be expressed as follows:

$$p_i\geq f_i\ p_n,\quad \text{for}\ i=1,\cdots,\ k, \tag{14}$$

$$p_i\leq f_i\ p_n,\quad \text{for}\ i=k+1,\cdots,\ n-1, \tag{15}$$

where f_i are constants. then we can show that the following theorem holds.

Theorem 2. If the utility function is additively separable and the price constraints are given by (14) and (15), then the market prices and shadow prices satisfy the following relations:

$$y'(b-z)=0, \quad (5)$$

$$p_i-y_i\geq 0 \text{ for } i=1,\cdots, k, \quad (16)$$

$$p_i-y_i \leq 0 \text{ for } i=k+1,\cdots, n-1, \quad (17)$$

$$(p_i-y_i)(p_i-f_i\, p_n)=0 \text{ for } i=1,\cdots, n-1, \text{ and} \quad (18)$$

$$\sum_{i=1}^{n} z_i p_i/e_i=\sum_{i=1}^{n} b_i y_i/e_i. \quad (19)$$

Proof. With the price constraints (14) and (15), equation (13) reduces to

$$w_i=(p_i-y_i)z_i/(p_i\, e_i), \text{ for } i=1,\cdots, k, \quad (20)$$

$$w_i=(y_i-p_i)z_i/(p_i\, e_i), \text{ for } i=k+1,\cdots, n-1, \text{ and} \quad (21)$$

$$y_n=p_n-(p_n e_n/z_n)\left\{-\sum_{i=1}^{k} w_i\, f_i+\sum_{i=k+1}^{n-1} w_i\, f_i\right\}. \quad (22)$$

And condition (7) can be expressed as follows:

$$w_i(p_i-f_i\, p_n)=0, \text{ for } i=1,\cdots, n-1. \quad \cdots(23)$$

Conditions (16) and (17) follow from conditions (20) and (21) and the nonnegativity property of w. Condition (18) follows from (20), (21) and (23). By substituting w_i in (20) and (21) into (22) and using (5) and (23), we can derive the condition (19). QED

This theorem says that if the market price of a commodity has a lower bound then the market price should be greater than or equal to the shadow price and there is a complementarity relation between the difference of the market price and shadow price and the difference of the market price and the lower bound. Similar relation holds also for the ceiling price case. These relations seem to be intuitively clear but these hold only for the consumers' market price not for the producers' market price. Kim (1984) derived a set of relations between shadow prices and producers' market prices.

Condition (19) implies that the weighted sum of the market values of consumed commodities should be equal to the weighted sum of the shadow values of initial resources. Especially, if the utility function is homogeneous with constant e_i, for example, the logarithmic utility function or the addi-log utility function with identical c_i, then e_i is identical for all commodities and can be deleted from the equation(19). In this case, the market value of all consumed commodities should be equal to the shadow value of initial resources.

5. Logarithmic Utility Function Case

Suppose that the utility function is given by

$$U(z)=\sum_{i=1}^{n} a_i \log z_i, \text{ where } a_i>0, \text{ and } \sum_{i=1}^{n} a_i=1. \quad (24)$$

Then we can show that the set of conditions, (1), (2), (5) and (14)$\cdots$(19) is equivalent to the following linear progrm (See Appendix for proof):

$$\text{Maximize} \quad \sum_{i=1}^{k} f_i\, z_i+z_n \quad (25)$$

subject to $(f_i/a_i)z_i-(1/a_n)z_n\leq 0,\ i=1,\cdots,\ k,$ (26)

$(1/a_n)z_n\leq f_i\, b_i/a_i,\ i=k+1,\cdots,\ n-1$ (27)

$0\leq z_i\leq b_i,\ i=1,\cdots,\ k$ and n. (28)

The amount of consumption, z_i, for $i=1$, k and n are given by the solution of the above linear program and z_i for $i=k+1$, $n-1$ are given by $z_i=b_i$ (See Appendix for proof).

Notice that since $p_i=a_i/z_i$ for $i=1,\cdots,\ n$ and $z_i=b_i$ for $i=k+1,\cdots,\ n-1$, the constraints (26) and (27) are nothing but the price constraints. Hence the linear program maximizes the objective function (25) subject to the material balancing constraints and the price constraints which are expressed in terms of the consumption levels.

We can find the solution of the linear program in a closed form as follows:

$$z_n=\text{Min}\ \{b_n, a_n f_i\, b_i/a_i \text{ for } i=k+1,\ n-1\},$$

$$z_i=\text{Min}\ \{b_i,\ a_i z_n/(f_i a_n)\},\ \text{for } i=1,\cdots,\ k.$$

The market price p_i can be found from the relation $p_i=a_i/z_i$.

Since z_i for $i=k+1$, $n-1$ are given by b_i, each ceiling price constraint provides an upper bound, $a_n f_i\, b_i/a_i$, on the level of consumption of the numeraire good. Hence z_n is determined by the minimum of these bounds and the initial endowment b_n. Then the price of the numeraire good is a_n/z_n and the minimum price constraint for i provides an upper, $a_i z_n/(f_i a_n)$, on the consumption level of that commodity. The minimum of this and the initial endowment is the solution.

The shadow price system y satisfies the following linear program:

Minimize $\sum_{i=1}^{n-1}(f_i b_i/a_i-b_n/a_n)b_i y_i$ (29)

Subject to $\sum_{i=1}^{n-1} b_i y_i\leq 1,$ (30)

$0\leq b_i y_i\leq a_i,$ for $i=1,\cdots,\ k,$ (31)

$b_i y_i\geq a_i,$ for $i=k+1,\cdots,\ n-1,$ (32)

The shadow price for the numeraire good, y_n, is given from the relation $\sum_{i=1}^{n} b_i y_i=1$ which is derived from (19). An optimal solution of this linear program is as follows:

(Case I) When $b_n\leq a_n f_i b_i/a_i$ for all $i=k+1,\cdots,\ n-1$.

$$y_i=\begin{cases}0, & \text{if } 1\leq i\leq k \text{ and } a_i/b_i\leq f_i(a_n/b_n),\\ a_i/b_i, & \text{otherwise.}\end{cases}$$

(Case II) When $b_n>a_n f_i\, b_i/a_i$ for some i between $k+1,\cdots,\ n-1$.

Let i^* be the index which minimizes $a_n f_i b_i/a_i$ for i between $k+1$ and $n-1$.

$$y_i=\begin{cases}0, & \text{if } 1\leq i\leq k \text{ and } a_i/b_i\leq f_i(a_n/b),\\ a_i/b_i, & \text{otherwise, except } i^*,\\ [a_{i^*}+a_n+\sum_{j=1}^{k}(a_j-b_j y_j)]/b_{i^*}, & \text{if } i=i^*.\end{cases}$$

6. A Graphic Example

To illustrate our results, let us consider the following example:

$n=2$, $U(z)= .5 \log z_1 + .5 \log z_2$, and $b'=(1,1)$.

Figure 1 is the commodity space showing the feasible region, OABC, and indifference curves. The optimal solution without any institutional price constraints is the point B. In this case, the market price is equal to the shadow price and $p_1=p_2= .5$.

Now suppose that the planner faces an institutional price constraints as follows:

$$p_2 \geq 2p_1. \quad \cdots(33)$$

Since the consumption behavior is guided by the market prices, the tangent line to the indifference curve represents the relative market prices of the two commodities. Only the

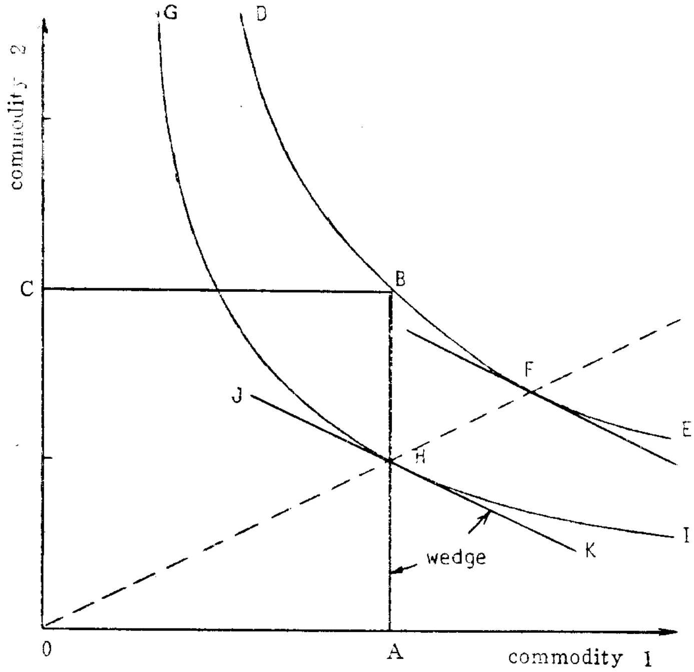

Figure 1. Commodity Space

segment FE has a tangent whose slope satisfies the constraint (33), where F is a point whose slope is such that $p_2=2p_1$. By tanslating the segment FE until it touches the feasible region, we obtain the optimal solution H. The market price supporting this solution is the slope of the tangent line to the point H. on the indifference curve GHI. On the other hand, the shadow price system is represented by the slope of the segment AB. Hence, the solution is $(z_1, z_2, p_1, p_2, u_1, u_2)=1, .5, .5, 1, 1, 0)$. Actually, this can be directly found from the results in section 5.

7. Conclusions

Our model is in a primitive stage to apply to a practical problem since it contains only the consumption side of the economy. Kim (1984) already developed a model which contains only production side of the economy with linear technologies. It would be an interesting work to develop a model with both consumption and production by combining these two models.

Appendix

If the utility function is given by (5.1), then $p_i=\partial U(z)/\partial z_i=a_i/z_i$. By eliminating p_i we can derive the following set of conditions for z and y:

$$b_i-z_i\geq 0,\ i=1,\cdots,\ n, \tag{34}$$

$$y_i(b_i-z_i)=0,\ i=1,\cdots,\ n, \tag{35}$$

$$-(f_i/a_i)z_i+(1/a_n)z_n\geq 0,\ i=1,\cdots k, \tag{36}$$

$$(f_i/a_i)z_i-(1/a_n)z_n\geq 0,\ i=k+1,\cdots,\ n-1, \tag{37}$$

$$a_i-b_iy_i\geq 0,\ i=1,\cdots k, \tag{38}$$

$$-a_i+b_iy_i\geq 0,\ i=k+1,\cdots,\ n-1, \tag{39}$$

$$(a_i-b_iy_i)\ [(f_i/a_i)z_i-(1/a_n)z_n]\ =0,\ i=1,\cdots n-1, \tag{40}$$

$$\sum_{i=1}^{n} b_iy_i=1, \tag{41}$$

$$y_i\geq 0,\ i=1,\cdots n, \tag{42}$$

Conditions (36) and (37) follow from (14) and (15), respectively. Conditions (16) and (17) conbined with (35) provide (38) and (39), respectively. Conditions (40) and (41) follow from (18) and (19), respectively.

Define $s_i=(f_i/a_i)\ (b_i-z_i)$ for $i=1,\ n-1$ and $s_n=(1/a_n)(b_n-z_n)$. Also define $t_i=a_i-b_iy_i$ for $i=1,\cdots,\ k$ and $t_i=-a_i+b_iy_i$ for $i=k+1,\cdots,\ n-1$. Since (39) implies that $y_i>0$ for $i=k+1,\cdots n-1$, (35) implies that $s_i=0$ for $i=k+1,\cdots,\ n-1$. Therefore $z_i=b_i$ for $i=k+1,\ n-1$. Define $g_i=(b_n/a_n-f_ib_i/a_i)$ for $i=1,\cdots,\ k$ and $g_i=(-b_n/a_n+f_ib_i/a_i)$ for $i=k+1,\cdots,\ n-1$. Then the system (34)—(42) reduces to a linear complementarity problem (q,M) with variable x defined in the follows:

$$x'=(s_1,\cdots,\ s_k, s_n, t_1,\cdots,\ t_{n-1}),$$

$$q'=(a_1,\cdots,\ a_k, a_n, g_1,\cdots g_{n-1}),$$

$$M=\begin{pmatrix}0 & A'\\ -A & 0\end{pmatrix},$$

$$A=\begin{pmatrix}-I_k & e_k\\ 0 & -e_{n-k-1}\end{pmatrix},$$

where I is the identity matrix and e is the column vector of unit elements. The subscripts denote the dimension of these matrix and vectors.

Because of the special form of the matrix M, this linear complementarity problem is equivalent to the following linear program:

(Problem A. 1)

$$\text{Maximize}\quad -\sum_{i=1}^{k} a_is_i-a_ns_n,$$

$$\begin{aligned}\text{subject to}\quad & s_i-s_n\geq -g_i,\ i=1,\cdots,\ k,\\ & s_n\geq -g_i,\ i=k+1,\cdots,\ n-1,\\ & s_i\geq 0,\ \text{for } i=1,\cdots,\ k \text{ and } n.\end{aligned}$$

The dual problem is

(Problem A. 2)

$$\text{Minimize} \quad \sum_{i=1}^{n-1} g_i t_i,$$

$$\text{subject to} \quad t_i \leq a_i, \quad i=1, \cdots, k,$$

$$\sum_{i=k+1}^{n-1} t_i - \sum_{i=1}^{k} t_i \leq a_n,$$

$$t_i \geq 0, \quad i=1, \cdots, n-1.$$

Substituting $s_i = (f_i/a_i)(b_i - z_i)$ into Problem A. 1, we can derive the linear program (25)—(28). Substituting $t_i = a_i - b_i y_i$ for $i=1, \cdots, k$ and $t_i = -a_i + b_i y_i$ for $i=k+1, \cdots, n-1$, we can derive the linear program (29)—(32).

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