論文 85-22-5-13

An Alternative Point-Matching Technique for Fredholm Integral Equations of Second Kind

(第2種 Fredholm 積分方程式의 새로운 數値解法)

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要 約

Fredholm 제2종 적분 방정식의 수치해법에 관한 새로운 기법을 제시하였다. 문제 영역의 절점에 데이터를 혼합 형태로 가함으로써 근사해를 구하였다. 수치 해법에서 오차를 줄이기 위하여 모든 절점에서 2번 연속 미분가능한 cubic B-spline 함수를 기저함수로 사용하였다. 기저함수로서 cubic B-spline 함수를 이용한 본 기법의 결과와 기저함수로 pulse 함수 test 함수로는 delta 함수를 이용한 모멘트법의 결과를 예제를 통하여 비교하였다. 또한 이 방법에 대한 수렴 조건을 기술 하였다.

Abstract

An alternative technique for the numerical solution of Fredholm integral equations of second kind is presented. The approximate solution is obtained by fitting the data in mixed form at knots in the region of the problem. To decrease the error in the numerical solution, cubic B-spline functions which are twice continuously differentiable at knots are employed as basis function. For a given example, the results of this technique are compared with those of Moment method employing pulse functions for basis function and delta functions for test function and found to be in good agreement.

I. Introduction

Numerical methods for solving the boundary-value problems and eigenvalue problems are employed in various electromagnetic field problems.

Differential, integral, and integro-differential equations can be solved with suitable numerical techniques.

Integral equations appear in their own right, and besides they often appear as alternative formulation of problems in differential equations.

Moment method which has many advantages, that is, simple software, short CPU time, and less core memory capability, has been extensively utilized to slove various field problems.

Usually, pulse expansion and delta testing functions are employed in this procedure, due to the simplicity of the resulting expressions.

The error resulting from the fact that the discrete information of knots are responsible for the entire information of the region of interest can be reduced by increasing the

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接受日字 : 1985年 3月 21日

number of knot points, but this can lower the speed of convergence to exact solution.[1,2]

To decrease the resulting error, various numerical techniques are considered.[3,4] With suitable expansion function, we can minimize the error.

Spline interpolation by means of the spline functions is a relatively new technique for field problems.[5,6]

Davis treated properties of spline function in solving electromagnetic field problem.[5]

Chua presented interpolation technique for CAD of microstrip by using cubic B-spline function.[6]

This paper presents an alternative point-matching technique for solving Fredholm integral equations of second kind, which arise from the ground wave propagation problems[7], the magnetic field integral equation formulations for a perfectly conducting scatterer and the integral equation formulations for the body of revolution with extended boundary condition [8], by using mixed point-matching technique, and used cubic B-spline function as basis function to reduce the error of the approximation.

We present the cases of convergence to the exact solution with respect to the length of elements and the coordinate of match points.

To prove the proposed algorithm, the results for a given problem of this algorithm are compared to those of Moment method.

II. Algorithm

Consider the following Fredholm integtral equation of second kind;

$$\tilde{u}(t)+\alpha \int_a^b k(t,s)u(s)ds = f(t) \qquad (1)$$

where u(t) is the unknown function to be approximated defined on the interval $T=[a,b]$, f(t) is a given excitation function on T, k (t,s) is a given kernel function which may or may not be singular, and α is a given constant. We can suppose an approximate solution of (1) over N-dimensional subspace as follow;

$$\tilde{u}(t) = \sum_{i=1}^{N} a_i \quad \psi_i(t) \qquad (2)$$

where $\psi_i(t)$ is i-th linear independent basis function and a_i is its coefficient to be obtained. In this paper, we use cubic B-spline function as basis function.

Thus the approximate solution (2) has error

$$e(t)=f(t)-\sum_{i=1}^{N} a_i [\psi_i(t)-\alpha \int_a^b k(t,s)\psi_i(s)\, ds] \quad (3)$$

We can apply to equation (3) with the condition $\frac{\partial e(t)}{\partial t} = 0$ in the interior region of T;

$$\sum_{i=1}^{N} a_i [\frac{\partial}{\partial t} \psi_i(t)-\alpha \int_a^b \frac{\partial}{\partial t} k(t,s)\, \psi_i(s)\, ds] = \frac{\partial f(t)}{\partial t} \qquad t \in (a,b) \qquad (4)$$

and with the condition e(t)=0 on the end points of T

$$\sum_{i=1}^{N} a_i [\psi_i(t)-\alpha \int_a^b k(t,s)\, \psi_i(s)\, ds] = f(t) \quad t=a,b \quad (5)$$

Combining equation (4) and (5), we can obtain matrix equation by using point matching technique at (N-2) knots in the interior region and 2 knots on end points of T

$$S\, D = L \qquad (6)$$

where

$$s_{ij}=[\frac{\partial}{\partial t}\psi_j(t)-\alpha \int_a^b \frac{\partial}{\partial t} k(t,s)\psi_j(s)ds]\,|_{t=t_i}\; t_i \in (a,b)$$

$$s_{ij}=[\psi_j(t)-\alpha \int_a^b k(t,s)\, \psi_j(s)\, ds]\,|_{t=t_i} \quad t_i = a,b$$

$$D = [a_1\ a_2\ a_3\ \ldots\ldots\ a_n]$$

$$\ell_i = \frac{\partial f(t)}{\partial t}\,|_{t=t_i} \quad t_i \in (a,b)$$

$$\ell_i = f(t)\,|_{t=t_i} \qquad t_i = a,b$$

From equation (2) and (6), we can obtain the approximated solution of the problem (1).

This procedure can be straightforwardly expanded to multidimensional field problem.

III. Cubic B-Spline Function

An extensive summary of its algebraic properties can be found in reference.[4]

On interpolation problem, global interpolation with a given set of data on the entire region give rise to 'Runge-Meray phenomenon which has a property of enormous error near the end point.

The error can be reduced by using piecewise polynomial and further reduced by fitting datas at Knots to be twice continuously differentiable.

B-spline function was first introduced by Schoenberg and widely used in curve fitting and surface fitting with advantages of low error and rapid convergence.

Cubic B-spline function satisfies smoothness constraints;

$$\tilde{u}^{(j)}(t_i)=u^{(j)}(t_i)\ j=0,1,2 \quad 1\leqslant i \leqslant N \qquad (7)$$

where superscript (j) is j-th derivative and t is the i-th knot point. To define the full set of cubic B-spline on T=[a,b], it is necessary to introduce six additional knots which are out side of T such that

$$t_1<t_2<t_3<t_4=a< \cdots\cdots$$
$$<b=t_{N-3}<t_{N-2}<t_{N-1}<t_N \qquad (8)$$

Thus, we can obtain unique interpolation function u(t) over N-dimensional subspace with N-3 elements.

Multi-dimensional cubic B-spline function can be obtained as a tensor product from the respective coordinates.

IV. Example And Computer Program

To verify the proposed algorithm, we choose the integral equation as follow;

$$u(t)-\alpha\int_{-0.5}^{0.5}k(t,s)\,u(s)\,ds=\sin t \qquad (9)$$

where

$$\alpha = 0.45$$
$$k(t,s) = \frac{1}{[(t-s)^2+1]^{3/2}}$$

The computed results of the example are given in table 1. The basis functions for the Moment method are the pulse function compared to the cubic B-spline function for the proposed technique.

The flowchart for the computer program is given in Fig. 1. The size of matrix is m*m for Moment method and (m+3)*(m+3) for the proposed technique with m elements.

The integration was performed by the 4-point Gauss-Legendre method with the single precision.

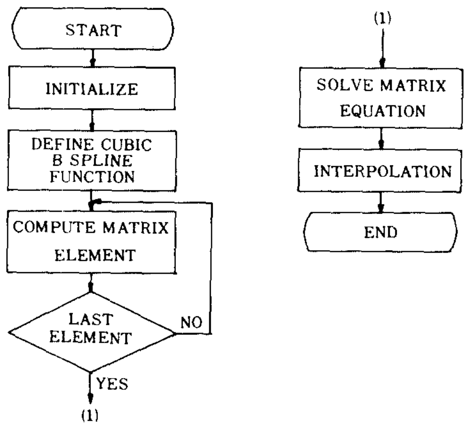

Fig. 1. Flowchart for the proposed technique.

Table 2 refers to the cases of convergence to the exact solution with respect to the Length

Table 1. Computed results.

method	number of elements	0	0.0	0.2	0.3	0.4	0.5
Moment	11	−2.711e-11	.09828	.19522	.28959	.38031	.46651
	41	1.397e-10	.10548	.20932	.30999	.40620	.49697
	91	6.668e-10	.10693	.21214	.31406	.41135	.50299
propesed technique	3	8.482e-8	.10812	.21441	.31728	.41553	.50799
	5	6.874e-7	.10811	.21443	.31737	.41555	.50792
exact		0	.10811	.21444	.31738	.41555	.50791

of elements and the position of match points.

Fig. 2 illustrate the element length and match point for the case of 2 in table 2.

Table 2. The cases of convergence.

case	element	match point
1	equal	equal
2	arbitrary	equal
3	equal	symmetric
4	arbitrary	symmetric
5	symmetric	symmetric

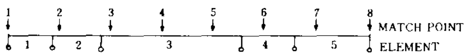

Fig. 2. Allocation of elements and match points.

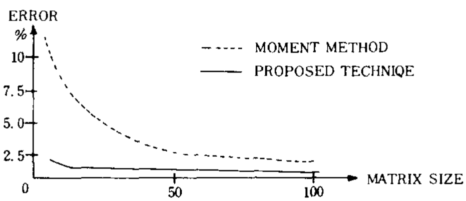

Fig. 3. % error v.s. matrix size.

V. Conclusion

A new algorithm for the numerical solution of Fredholm integral equations of second kind is proposed.

For a given example, the computed results of this technique are compared to the those of Moment method employing pulse and delta function for basis and test function, respectively.

The following items are verified.

1. Reduction of CPU time
2. Low memory capability
3. Rapid convergency

The possible cases of convergence of this technique are summalized with respect to the element length and the position of match points.

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