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Optimization of Weighting Matrix Selection

(狀態 比重 行列의 選擇에 대한 最適化)

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要 約

이 논문에서는 상태비중 행렬의 선택을 최적화하는 기법이 제시된다. 이 상태 비중 행렬은 폐루프 시스템의 응답특성이 이상적인 모델의 응답특성과의 차이가 최소화 되도록 선택되며 제시된 알고리즘의 타당성을 보이기 위하여 수치적인 예가 주어진다.

Abstract

A method optimizing selection of a state weighting matrix is presented. The state weighting matrix is chosen so that the closed-loop system responses closely match to the ideal model responses. An algorithm is presented which determines a positive semidefinite state weighting matrix in the linear quadratic optimal control design problem and an numerical example is given to show the effect of the present algorithm.

I. Introduction

In the linear quadratic regulator problem, so little is known about the relationships between the weighting matrices and various design specifications. Therefore the designer must resort to trial and error iterations. To deal with these limitations, various intuitive ways to select quadratic weighting matrices have been devised such as procedure for asymptotically placing desired closed-loop eigenvalues and eigenvectors [1-2] and various versions of model following [3-4]. As another indirect method, a time-weighted quadratic performance index can be used to give a response having a small overshoot and adequately damped without complicating weighting matrix selection [5-6].

In this paper, a new weighting matrix selection algorithm is presented. To translate design specifications, a state weighting matrix is optimized so that the closed-loop system responses resulting from the Riccati equation closely match to the model responses. An algorithm is presented which determines a positive semidefinite state weighting matrix in the linear quadratic optimal control design problem and an numerical example is given to show the effect of the present algorithm.

II. Problem statement and solution

Let the linear time-invariant system be defined by

$$x(t) = Ax(t) + Bu(t)\ , \ x(0) = x_o \tag{1-a}$$

$$y(t) = Cx(t) \tag{1-b}$$

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where x is an n-dimensional state vector, u is an m-dimensional control vector, and y is a r-dimensional vector of responses. The constant matrices A, B and C are of appropriate dimensions. In the following, the prime will denote the transpose and tr[M] the trace of the matrix M.

Let the optimal feedbak gain K minimizes the following quadratic performance index

$$J_1 = \frac{1}{2} \int_0^\infty (x'Qx + u'Ru)\, dt \quad t \qquad (2)$$

where the matrices Q and R are symmetric positive semidefinite and positive definite matrices such that $Q = D'D$, D is an nxn matrix. The matrix pairs (A, B) and (A, D) are assumed to be completely controllable and observable, respectively. Controllability of the matrix pair (A, B) ensures boundness of the cost J_1 in (2) and observability of (A, D) ensures that the closedloop system resulting from the Riccati matrix equation is asymtotically stable. Without loss of generality, we assume that the weighting matrix of the control signal R is identity matrix [7]. Then the feedback gain K satisfies the following relation:

$$K = - B'P \qquad (3)$$

where the matrix P is the solution of the Riccati matrix equation.

$$PA + A'P + D'D - PBB'P = 0 \qquad (4)$$

The dynamics of some ideal systems, which is called the model, may be described by the equation

$$\dot{x}_m(t) = A_m x_m(t) \qquad (5)$$

where x_m is a r-dimensional model state vector.

The problem is to choose the positive semidefinite state weighting matrix Q so that the closed loop system responses closely match to the model responses. This can be achieved by minimizing the peformance index

$$J(D) = \int_0^\infty (\dot{y} - A_m y)'W (\dot{y} - A_m y)\, dt. \qquad (6)$$

This performance index penalizes the difference between the derivatives of actual system response y and those derivatives that would arise if the system were identical to the model. The performance index (6) which includes an implicit model-following term may be replaced by

$$\hat{J} = \int_0^\infty (y - x_m)'W(y - x_m)\, dt \qquad (7)$$

which involves the concept of the virtual model-following [8]. A disadvantage of the approach baced on the performance index (7) is that it leads to an increse in the dimension of the problem. Eqn. (1-a) and eqn. (3) together with u = Kx yield

$$\dot{x} = (A - BB'P)x = Fx\ ; \qquad (8)$$

where

$$F = (A - BB'P)\ \cdot$$

The cost J can be expressed as [9]

$$J = tr\,(P_1 X_0) \qquad (9)$$

where P_1 is the solution of the following Lyapunov matrix equation

$$F'P_1 + P_1 F + (CF - A_m C)'W \cdot (CF - A_m C) = 0 \qquad (10)$$

and $X_0 = x_0 x_0'$. To determine the matrix D which minimizes the cost J subject to the constraints in eqn. (4) and (10), we use the Lagrangian multiplier approach [10]. Then the Hamiltonian for this problem is expressed as

$$\begin{aligned} H(D,P,L,P_1,L_1) &= tr\,(P_1 X_0) \\ &+ tr\left\{ L_1'[F'P_1 + P_1 F + (CF - A_m C)'W (CF - A_m C)]\right\} \\ &+ tr\,[L'(A'P + PA + D'D - PBB'P)] \end{aligned} \qquad (11)$$

where L_1 and L are the symmetric Lagrange multiplier matrices. The necessary conditions for the solution are derived by taking the partial derivatives of (11) with respect to D, P, P_1, L_1 and L, and equating them to zero. The necessary conditions to be satisfied by the matrix D which minimizes the performance index (6) are given by

$$\frac{\partial H}{\partial D} = 2DL = 0.. \tag{12}$$

Here the matrix L satisfies the following lyapunov equations:

$$FL + LF' + Y + Y' = 0 \tag{13}$$

$$FL_1 + L_1 F' + X_o = 0 \tag{14}$$

where $Y = - BB' [P_1 + C'W(CF - A_m C)] L_1$.

III. A Computational Algorithm

To obtain the optimal solution, one can use the following iterative algorithm

(1) Select any initial matrix D.

(2) Determine $\frac{\partial H}{\partial D}$ from eqn. (12).

(3) If $\frac{\partial H}{\partial D}$ satisfies convergence criteria, iteration is completed. Otherwise find a new value of the matrix D using any gradient based method [11].

(4) Return to the step (2).

IV. Example

Consider a third-order system to control position of permanent magnet DC motors as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.02 & 0.83 \\ 0 & -3481 & -439 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 9259 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The weighting matrices and initial conditions also are chosen as W=I, $x(0)=[-1\ 0\ 0]'$ and $x_m(0)=[-1\ 0]'$.

Case 1

When the ideal model dynamics is chosen as follows

$$A_m = \begin{bmatrix} 0 & 1 \\ -25 & -7.07 \end{bmatrix},$$

eqn. (12) gives rise to

$$Q = \begin{bmatrix} 225 & 0 & -0.03 \\ 0 & 0.01 & 0 \\ -0.03 & 0 & 0.243 \end{bmatrix},$$

$$K = [15 \quad 3.875 \quad 0.449], \quad J \simeq 0 .$$

Therefore the responses of the closed-loop system are equal to those of the model.

Case 2

When the model dynamics is chosen as

$$Am = \begin{bmatrix} -1 & 1 \\ -25 & -7.07 \end{bmatrix},$$

similarity we can obtain

$$Q = \begin{bmatrix} 100 & 0 & 0.018 \\ 0 & 0.01 & 0 \\ 0.018 & 0 & 0.107 \end{bmatrix},$$

$$K = [10 \quad 2.458 \quad 0.283], \quad J = 0.316.$$

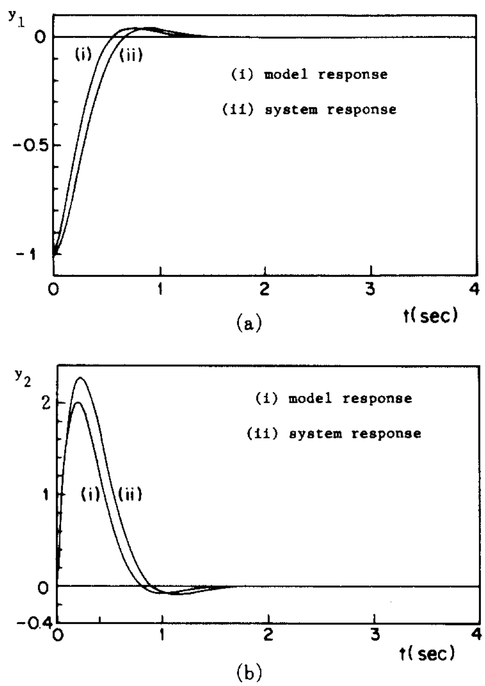

Fig. 1. Responses of model and system.

Then, the transient responses are plotted in Fig. 1.

V. Conclusions

A new weighting matrix selection algorithm has been presented. To translate design specifications, a state weighting matrix is optimized so that the closed-loop system responses resulting from the Riccati matrix equation closely match to the model responses. An algorithm has been presented which determines a positive semi-definite state weighting matrix in the linear quadratic optimal control design problem and an numerical example has been given to show the effect of the present algorithm.

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