論 文
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可觀測的인 랜덤 係數를 가진 스토캐스틱 시스템의 最適制御

Optimal Control of Stochastic Systems with Completely Observable Random Coefficients

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요 약

본 論文은 랜덤 係數들을 가진 線型시스템의 制御에 관하여 研究하였다. 評價函數는 2次型이고 랜덤 係數들은 완전히 觀測된다고 가정한다. 문제를 체계화하는데 있어 스토캐스틱 프로세스는 스토캐스틱 微分方程式에 일치하는 일위적 強한 解가 存在함을 증명한다. 最適制御를 위한 條件은 주어진 非線型偏微分方程式에 대한 Cauchy문제로 그 解의 存在를 통하여서만 표현된다. 最適制御法則은 狀態變數에 대해서 線型이고 랜덤 파라메타의 非線型 函數로 구해진다.

Abstract

The control of a linear system with random coefficients is discussed here. The cost function is of a quadratic form and the random coefficients are assumed to be completely observable by the controller. Stochastic Process involved in the problem by the controller. Stochastic Process involved in the problem formulation is presented to be the unique strong solution to the corresponding stochastic differential equations. Condition for the optimal control is represented through the existence of solution to a Cauchy problem for the given nonlinear partial differential equation. The optimal control is shown to be a linear function of the states and a nonlinear function of random parameters.

1. Introduction

In some sense, every known deterministic mathematical model can be considered as the simplification of a suitable stochastic model. It may be of interest in general to study stochastic functional equations. Stochastic models based on the white noise may be given by the following ordinary differential equations:

$$\frac{dx(t)}{dt} = g(t, x(t), u(t)) + h(t, x(t))\, v(t), \tag{1-1}$$

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where $v(t)$ is a well-known white noise. The formal solution of (1-1) is given by

$$x(t) = x_0 + \int_0^t g(s, x(s), u(s))ds + \int_0^t h(s, x(s))v(s)ds, \tag{1-2}$$
$$x(0) = x_0.$$

The white noise is the formal derivative of Brownian motion w(t)

$$v(t) \approx \frac{dw(t)}{dt}. \tag{1-3}$$

The equation (1-1) is formally equivalent the differential form

$$dx(t) = g(t, x(t), u(t))\, dt + h(t, x(t))dw(t). \tag{1-4}$$

The equation (1-4) is meaningful only in sofar as its integral representation

$$x(t)=x_0+\int_0^t g(s,x(s),u(s))ds + \int_0^t h(s,x(s))dw(s). \qquad (1\text{-}5)$$

In the normal sense, the equation (1-5) does not make any sense, because the last integral in (1-5) can not be defined by the usual Lebsgue-Stieljes sense. A first step toward the development of a theory of stochastic differential equations might therefore be a definition of a stochastic integral[1), 2)]. Let $x(t)$ be a stochastic process satisfying (1-5) for all $t\in[0,T]$. Then we say that $x(t)$ is generated according to a stochastic differential equation (1-4).

Most of the results in stochastic dynamic control are due to[3)-7)]. In regard to problems with partial observation the best results in Wonham's formulation of separation principle[3)] using stochastic Bellman dynamic programming. Dynamic programming is a useful approach in stochastic control. However, these conditions under dynamic programming are so much weak than those required in the deterministic case.

A typical example is a stochastic control system with random operated-valued coefficients[8)-11)]. This paper presents the problem of the optimal control of the stochastic differential equation with random matrix-valued coefficients. Sufficient conditions for an optimal control are expressed through the existence of bounded solution to a certain Cauchy problem for both Brownian motion and random process disturbances is dealt with in this study and these also include stochastic bilinear systems[12)].

2. Existence and Uniqueness of Solution

Let $[\Omega, F, P]$ be a complete probability space, F_t, an increasing family of sub-σ-algebra of F, $t\in[0,T]$, $\{x_t\}$, an F_t adapted process, and let $w_t^1=\{w_1(t), F_t\}$ and $w_t^2=\{w_2(t), F_t\}$ be independent of the Wiener processes. The random variable $x_1(0)$ and $x_2(0)$ are assumed to be independent of the Wiener processes w_t^1 and w_t^2 of dimension ℓ_1 and ℓ_2. Let $x_1(t)$ and $x_2(t)$, $t\in[0,T]$ be observable and continuous process of the controlled diffusion type with

$$dx_1(t)=A(t,x_2(t))x_1(t)dt+B(t,x_2(t))u(t)dt+G(t,x_2(t))dw_t^1 \qquad (2\text{-}1)$$

$$dx_2(t)=C(t,x_2(t))dt+D(t,x_2(t))dw_t^2 \qquad (2\text{-}2)$$

$$x_3(o)=x_{1_0},\ x_2(o)=x_{2_0}.$$

Each of the measurable functional $A(t,\theta)$ $B(t,\theta)$, $G(t,\theta)$, $C(t,\theta)$, and $D(t,\theta)$ is assumed to be nonanticipative and have the dimension $n\times n$, $n\times p$, $n\times\ell_1$, $m\times m$, $m\times\ell_2$, respectively. It means Y_t-measurable where Y_t is the σ-algebra in the continuous finite space C_T of the continuous functions $\theta=\{\theta(s), s\leqslant T\}$ generated by the function θ_s, $s\leqslant t$.

The control $u(t)$ is a feedback of the current state. The problem is to find a control in an admissible control set U that minimizes the average cost functional

$$J(u)=E\{\int_0^T L(t,x(t),u(t))dt\}, \qquad (2\text{-}3)$$

where $x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$ and the response to the control $u(t)$.

The existence and uniqueness of the solution of (2-1) are given in the following assumptions [7),10),12)]. For each $\xi,\eta\in C_T$,

i) $u(t)\in U$, $B(t,\xi)u(t)\in[0,T]\times\mathbf{R}^m\times U$ where U is compact.

ii) $\int_0^T[|A(t,\xi)|+|C(t,\xi)|+|B(t,\xi)|+G^2(t,\xi)+D^2(t,\xi)]dt<\infty$. (2-4)

Along with (2-4) assuring the existence of the integrals in (2-1) and (2-2), we will also held the following assumptions.

iii) $\int_0^T C^2(t,\xi)dt<\infty$, $\inf D(t,\xi)\geqslant K_1$, $t\in[0,T]$, $\xi\in C_T$,

iv) $|A(t,\xi)-A(t,\eta)|^2+|B(t,\xi)-B(t,\eta)|^2 + |G(t,\xi)-G(t,\eta)|\leq K_2\int_0^t|\xi_s-\eta_s|^2 \cdot dM(s)+K_3|\xi_t-\eta_t|^2$,

$|C(t,\xi)-C(t,\eta)|^2+|D(t,\xi)-D(t,\eta)|^2 \leq K_4\int_0^t|\xi_s-\eta_s|^2dM(s)+K_5|\xi_t-\eta_t|^2$,

v) $A(t,\xi)^2+B(t,\xi)^2+G(t,\xi)^2\leq K_2\int_0^t(1+\xi_s^2)\cdot dM(s)+K_3(1+\xi_t^2)$,

$C(t,\xi)^2+D(t,\xi)^2 \leq K_4\int_0^t(1+\xi_s^2)dM(s)+K_5(1+\xi_t^2)$,

where $M(s)$ is a nondecreasing right continuous function, with $0\leqslant M(s)\leqslant 1$ and K_i, $i=1,2,\dots,5$, are positive constants.

We consider the following equations of the solutions of (2-1) and (2-2),

$$x_1(t) = x_{1_0} + \int_0^t A(s, x_2(s))x_1(s)\,ds + \int_0^t B(s,x_2(s))u(s)\,ds + \int_0^t G(s, x_2(s))\,dw_s^2 \quad (2\text{-}5)$$

and

$$x_2(t) = x_{2_0} + \int_0^t C(s, x_2(s))ds + \int_0^t D(s, x_2(s))\,dw_s^2, \quad (2\text{-}6)$$

and we will see the existence of strong solutions and uniqueness of (2-5) and (2-6).

Theorem 2.1

Let the nonanticipative functional $A(s, \eta)$, $B(s, \eta)$, $G(s, \eta)$, $C(s, \eta)$, $D(s, \eta)$, $s \in [0, T]$ satisfy the assumptions iv) and v), and let $|A(t,\eta)| \geqslant K_6 < \infty$, $B(t, \eta) \leqslant K_7 < \infty$. Then if $\begin{bmatrix} x_{1_0} \\ x_{2_0} \end{bmatrix}$ is an F_0-measurable random variable $E\begin{bmatrix} x_{1_0}^2 \\ x_{2_0}^2 \end{bmatrix} < \infty$. The stochastic differential equations (2-1) and (2-2) have a unique strong solution.

3. Optimal Control with the Complete Data

We consider the problems of optimal control for solutions of the stochastic differential equations based on complete data and fixed finite interval of time $[0, T]$. Consider again the following stochastic differential equations

$$dx_1(t) = A(t, x_2(t))\,x_1(t)\,dt + B(t, x_2(t))u(t)\,dt + G(t, x_2(t))\,dw^1, \quad (3\text{-}1)$$

$$dx_2(t) = C(t, x_2(t))\,dt + D(t, x_2(t))\,dw^2,\ x_1(0) = x_{1_0} \in \mathbf{R}^n,$$

$$x_2(0) = x_{2_0} \in \mathbf{R}^m,$$

where all dimensions are the same as the solutions (2-1) and (2-2).

The problem is to choose a control law $u(t)$ so as to minimize the cost functional

$$J(u) = E\left[\int_0^T L(t, x(t), u(t))dt\right].$$

For a solution of the optimal control we will introduce the following assumptions.

1) The admissible controls consist of $u(t)$: $t \in [0, T] \times \mathbf{R}^n \to U$ where U is a fixed compact convex each $0 \leqslant t' \leqslant T$, $u(t)$ is uniformly Hölder continuous with exponent $0<q<1$ and uniformly Lipschitz continuous in x on $t' \in [0, T]$ such that

$$|\, u(t', x) - u(t, x)\,| \leqslant C_1\, |t'-t|^q,$$
$$|\, u(t, x') - u(t, x') \leqslant C_2\, |x'-x|,$$

for some positive constants C_1 and C_2 for all x, $x' \in \mathbf{R}^{n+m}$, $t, t' \in [0, T]$.

Let H be a separable Hilbert space. For $t \in [0, T]$, $\eta \in \mathbf{R}^m$,

2) $A(t, \eta)$ is a family of random variable η_t such that $A(t, \eta_t)$ is Borel measurable.

3) $|A(t, \eta_t)| + |B(t,\eta_t)| + |\,G(t, \eta_t)| + |C(t, \eta_t)| + |D(t, \eta_t)| \leqslant K < \infty$.

4) The nonanticipative functional $A(t, \eta_t)\ \xi_t$ satisfies theroem 2-1 in given the limitation.

5) w_t is an H-valued Wiener process with the usual properties.

6) $E[x_0^2] < \infty$, x_0 is independent of w_t.

Let

$$L(t, x(t), u(t)) = x_1^*(t)Q(t, x_2(t))x_1(t) + u^*(t)R(t, x_2(t))u(t)$$

where * denotes the transpose of the given matrix or vectors, and then

7) $Q(t, \eta_t)$ is nonnegative definite matrices, and $R(t, \eta_t)$ is uniformly positive definite, i.e., elements of its inverse are uniformly bounded measurable function.

8) The control $u(t) \in U$ satisfies

$$\int_0^T E[\|\, u(t)^4 \,\|]\,dt < \infty,$$

and are such that (2-1) has a unique solution.

Let $s \in [0, T]$ be the initial time; $x_{1_s} = x_{1_0}$, the initial state, $u(t) \in U$, and $x_1(t)$, the corresponding response of the system (2-1). The conditional remaining cost on the time s=o is defined by

$$W^u(s, x_1, x_2) = E\left[\int_0^T L(t, x_1(t), u(t), dt \mid x_{1_s} = x_{1_0}, x_{2_s} = x_{2_0}\right], \quad (2\text{-}3)$$

as the expected cost corresponding to the control $u(t)$ and initial x_{1_0} and x_{2_0}. Here, T is fixed terminal time. The problem is to minimize $J(u)$ on U.

Let

$$\alpha(t, \eta_t) \triangleq \begin{bmatrix} G(t,\eta_t)G^*(t,\eta_t) & G(t,\eta_t)D^*(t,\eta_t) \\ D(t,\eta_t)G^*(t,\eta_t) & D(t,\eta_t)D^*(t,\eta_t) \end{bmatrix}$$

and assume that $\alpha(t, \eta_t)$ is uniformly positive definite over $t \in [0, T]$, $\eta_t \in R^m$, i.e.,

$$\sum_{i,j}^{n+m} \alpha_{ij}(t,\eta_t) y_i y_j \geq K|yy^*|, \quad K>0,$$

for all $y \in R^{n+m}$.

This essentially states that noise enters every component of (3-1), whatever the coordinate system.

Define

$$V(t,y) = \inf_u V^u(t, y),$$

where $V^u(t, y) = W^u(s, x_1(t), x_2(t)), y = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, (t,y) \in [0,T] \times R^{n+m}$.

From the assumptions it follows that[5]

$$V_s + A^u(s)V + L(s, y, u(s, y)) \geq 0,$$

where

$$A^u(s)V = \frac{1}{2} \sum_{i,j=1}^{n+m} \alpha_{ij} \frac{\partial^2}{\partial y_i \partial y_j^*} V + \sum_{i=1}^{n} (A(s,\eta_s)\xi + B(s,\eta_s)u_s)_i \frac{\partial}{\partial y_i} + \sum_{i=n+1}^{n+m} (C(s,\eta_s))_j \frac{\partial}{\partial y_i} V, \quad \eta_s \in R^m, \ \xi \in R^n,$$

and

$$V_s \triangleq \frac{\partial V}{\partial s}.$$

The above equaility holds if $u(t) = u^0(s,y)$, where $u^0(s,y)$ is an optimal feedback control law. this leads to the continuous time dynamic programming equation:

$$V_s + \min_u (A^u(s) V + L(s,y,u)) = 0, \qquad (3\text{-}4)$$
$$V(T, y) = 0$$

Theorem 3.1

Assume that the value function V satisfying the stochastic Bellman equation (3-4) exists and is differentiable in (t, y). If a control $u_t^0 \in U$ satisfies

$$A^{u^0}(t)\, V + L(t,y,u^0(t,y)) \leqslant A^u(t)\, V + L(t,y,u(t,y)), \qquad (3\text{-}5)$$

for all $u_t = u(t) \in U$, $(t,y) \in [0, T] \times R^n \times R^m$, then u_t^0 is an optimal control.

Theorem 3.2

The optimal u_t^0, $t \in [0, T]$, exists and is given by

$$u_t^0 = -R^{-1}(t, x_2(t))B^*(t, x_2(t))[\Lambda_1(t, x_2(t))x_1(t) + \frac{1}{2}\Lambda_2(t, x_2(t))], \qquad (3\text{-}6)$$

If there exist the nonnegative definite symmetric matrices Λ_1 and Λ_2 satisfying the following nonlinear partial differential equation, (3-7)

$$\dot\Lambda_1 + A^*\Lambda_1 + \Lambda_1 A + Q - \Lambda_1 BR^{-1}B^*\Lambda_1 + C^*\frac{\partial}{\partial x_2(t)}\Lambda_1 + \frac{1}{2}tr(DD^*\frac{\partial^2}{\partial x_2^*(t)\partial x_2(t)})\Lambda_1 = 0,$$

$$\dot\Lambda_2 + \Lambda_2 A - \Lambda_2 BR^{-1}B^*\Lambda_1 + C^*\frac{\partial}{\partial x_2(t)}\Lambda_2 + \frac{1}{2}tr(DD^*\frac{\partial^2}{\partial x_2(t)\partial x_2^*(t)})\Lambda_2 + 2(GD^*\frac{\partial}{\partial x_2(t)})^*\Lambda_1 = 0,$$

$$\Lambda_1(T, x_2(T)) = 0, \quad \Lambda_2(T, x_2(T)) = 0, \quad x_2(t) \in \boldsymbol{R}^m,$$

where the argument $(t, x_2(t))$ is omitted for brevity. The solution $\Lambda_1(t, \eta_t)$ and $\Lambda_2(t, \eta_t)$ to the above Cauchy problem can be shown to nonnegative definite and uniformly bounded for all $(t, \xi_t) \in [0, T] \times \boldsymbol{R}^m$[7].

Mohler depeoped stochastic bilinear systems are the diffusion models for migration of people, biological cell, etc.,[12]. The system equation in (3-1) is a class of coupled stochastic bilinear equations. In this particular case, the optimal control of the stochastic bilinear system of diffusion process (3-1) is given by (3-6). The following examples belong to class of coupled stochastic bilinear systems. If $A(t, x_2(t)) = 0$, $G(t, x_2(t)) = G(t)$, then the equation (3-1) is

$$dx_1(t) = B(t, \cdot)\, u(t)dt + G(t)\, dw(t), \quad x_1(0) \in \boldsymbol{R}^n, \qquad (3\text{-}8)$$

where $B(t,\cdot)$ is composed of unknown coefficients. Such uncertain parameter may be regarded as additional state variables. These additional state variables with uncertain gain may be approximated by

$$dx_2(t) = C(t, x_2(t))\, dt + D(t)\, dw(t), \quad x_2(o) \in \boldsymbol{R}^n. \qquad (3\text{-}9)$$

If $B(t,\cdot) = B(t)\, x_2(t)$, (3-8) is bilinear in $x_2(t)$ and $u(t)$, and the system has extended state with $\boldsymbol{R}^{n+m}$. At this point the problem of uncertain parameter becomes a parameter identification problem and the system equation is a stochastic bilinear differential equation. An aircraft landing process[11] may be represented by this type of stochastic bilinear equation.

4. Simulation Results

Consider the following stochastic differential equation

$$dx(t) = (3.75 + 1.5 \tan^{-1} z(t))\, dt + u(t)dt,$$
$$d\, z(t) = dw(t),\ x(o) = 1.0 \qquad (3\text{-}10)$$

where $x(t)$ is observable and $w(t),\ t \in [0,1]$ is a Wiener process. The solution to the optimal-control problem yields a control $u^\circ(t)$ that minimizes the criterion

$$J(u) = E\int_0^T (x(t)^2 + u^2(t))\, dt.$$

According to the results discussed in section 3, the stochastic control problem in (3-6) has the solution of the form

$$u^0{}_t = -\Lambda\,(t, z(t))\, x(t),$$
where $\Lambda\,(t, \xi),\ \ \xi \in \boldsymbol{R}^1$, satisfies

$$\Lambda(1, \xi) = 0,\ t \in [0, 1].$$

Figure 1 shows optimal control and suboptimal control for (3-10) with $(3.75 + 1.5 \tan^{-1} z(t))$ replaced by

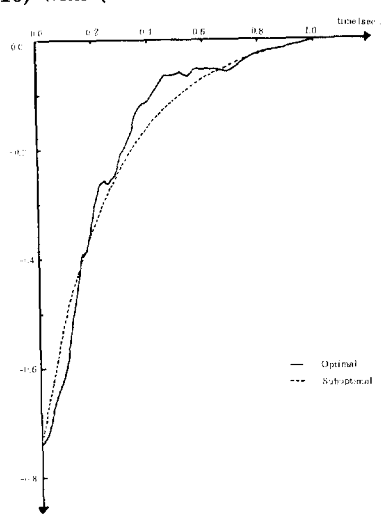

Fig. 1. Optimal control law and suboptimal control for equation (3-10).

$E[3.75 + 1.5 \tan^{-1} z(t)]$. Figure 2 shows the sample paths according to optimal control and suboptimal control in (3-10).

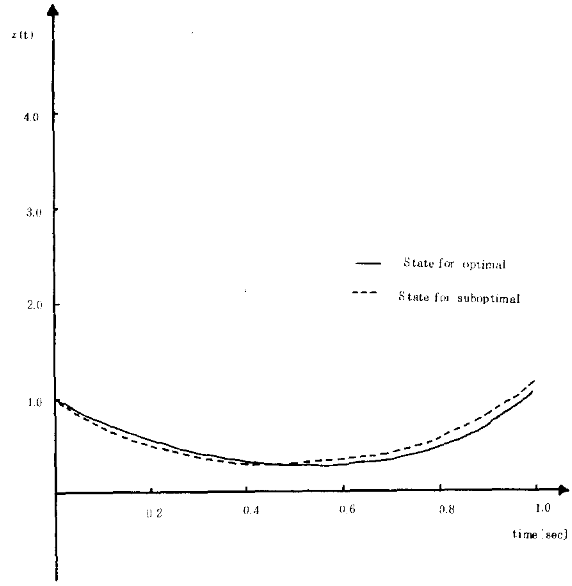

Fig. 2. Realization of x(t) under optimal and suboptimal for equation (3-10).

5. Conclusion

The optimal control for a certain linear system with random coefficients is studied. The stochastic differential equation (2-1) is described by control systems which are linear in the observable part of the state variables and nonlinear, in a very general, functional manner in random coefficients and control variables.

In the section 2 the existence and uniqueness of a solution to (2-1) and (2-2) is studied here. Sufficient conditions for an optimal control are expressed through the existence of a bounded solution to a certain Cauchy problem for prabolic type of partial differential equations. For the quadratic cost function the explicit formulae describing the control law is derived.

Simulation results shown in figure 1 present that the optimal control law is more complete them the suboptimal cases with $E[3.75 + 1.5 \tan^{-1} z(t)]$.

The proofs of the theorems in the paper are omitted for brevity.

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