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A NOTE ON WITT RINGS OF 2-FOLD FULL RINGS

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1. Introduction

D.K. Harrison [5] has shown that if R and S are fields of characteristic different from 2, then two Witt rings $W(R)$ and $W(S)$ are isomorphic if and only if $W(R)/I(R)^3$ and $W(S)/I(S)^3$ are isomorphic where $I(R)$ and $I(S)$ denote the fundamental ideals of $W(R)$ and $W(S)$ respectively. In [1], J.K. Arason and A. Pfister proved a corresponding result when the characteristics of R and S are 2, and, in [9], K.I. Mandelberg proved the result when R and S are commutative semi-local rings having 2 a unit. In this paper, we prove the result when R and S are 2-fold full rings.

Throughout this paper, unless otherwise specified, we assume that R is a commutative ring having 2 a unit. A quadratic space (V, B, ϕ) over R is a finitely generated projective R-module V with a symmetric bilinear mapping B: $V\times V\to R$ which is nondegenerate (i.e., the natural mapping $V\to \mathrm{Hom}_R(V,R)$ induced by B is an isomorphism), and with a quadratic mapping $\phi: V\to R$ such that $B(x, y)=(\phi(x+y)-\phi(x)-\phi(y))/2$ and $\phi(rx)=r^2\phi(x)$ for all x, y in V and r in R. We denote the group of multiplicative units of R by $U(R)$. If (V, B, ϕ) is a free rank n quadratic space over R with an orthogonal basis $\{x_1, \cdots, x_n\}$, we will write $\langle a_1, \cdots, a_n\rangle$ for (V, B, ϕ) where the $a_i=\phi(x_i)$ are in $U(R)$, and denote the space by the table $[a_{ij}]$ where $a_{ij}=B(x_i, x_j)$. In the case $n=2$ and $B(x_1, x_2)=1/2$, we reserve the notation $[a_{11}, a_{22}]$ for the space.

2. The quotients $I(R)/I(R)^2$ and $U(R)/U(R)^2$

Let G be a muitiplicative abelian group of exponent 2 with identity e. We will write $\boldsymbol{Z}[G]$ for its group ring and $\{g\}$ for the image in $\boldsymbol{Z}[G]$ of an element g of G. Let M be the kernel of the ring homomorphism $\boldsymbol{Z}[G]\longrightarrow\boldsymbol{Z}\longrightarrow\boldsymbol{Z}/2\boldsymbol{Z}$ defined by sending each group element to 1 and reducing mod $2\boldsymbol{Z}$. Then M consists of elements of the form $\sum n_i\{g_i\}$ with $\sum n_i$ even, and is generated additively by the elements $\{e\}+\{g\}$. We define $\mathrm{d}_g: M\longrightarrow G$ by

$$\sum n_i\{a_i\}\longmapsto (\prod a_i^{n_i})\cdot g^{\Sigma n_i/2}$$

where $g\in G$. It is clear that d_g is a group homomorphism. If K is an ideal of $\boldsymbol{Z}[G]$ contained in M we will write $\overline{M}$ for the ideals M/K in $\boldsymbol{Z}[G]/K$.

LEMMA 2.1. *Let K be an ideal of $\boldsymbol{Z}[G]$ contained in M with $\{e\}+\{g\}$ in K and*

$d_g(K) = e$. *Then* d_g *induces an isomorphism* $\overline{M}/\overline{M}^2 \longrightarrow G$ *of groups with the inverse isomorphism is given by* $b \longrightarrow \mathrm{cl}(\{e\}+\{bg\})$ *where* cl *denotes the canonical map* $M \longrightarrow \overline{M} \longrightarrow \overline{M}/\overline{M}^2$.

Proof. It is just [9, Lemma 3.1, p.524].

A commutative ring R having 2 a unit is called n-fold full [7, p.149] if for every $n\times 3$ matrix $A=[a_{ij}]$ with unimodular rows there is an element w in R such that

$$A\begin{pmatrix} 1 \\ w \\ w^2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix}$$

where $v_1, v_2, \cdots, v_n$ are units. Thus an n-fold full ring is k-fold full for $1 \leq k \leq n$.

We now specialize to the case where G is the group $U(R)/U(R)^2$ for an 1-fold full ring R. A cap will be used to indicate reduction mod $U(R)^2$

THEOREM 2.2. *Let R be an 1-fold full ring. Then there is an abelian group isomorphism from* $I(R)/I(R)^2$ *onto* $U(R)/U(R)^2$ *which is given by* $[\langle a_1, \cdots, a_n\rangle] + I(R)^2 \longmapsto (\prod \hat{a}_i)\cdot(\widehat{-1})^n$ *with inverse* $\hat{a} \longmapsto [\langle 1, -a\rangle] + I(R)^2$.

Proof. Let $G=U(R)/U(R)^2$, then the ring homomorphism which takes $\{\hat{a}\} \longmapsto [\langle a\rangle]$ is a surjection of $\boldsymbol{Z}[G]$ onto $W(R)$ [7, p.151] whose kernel we denote by K. The ideal K is generated by $\{\hat{1}\}+\{\widehat{-1}\}$ and the elements of the form $\sum_{i=1}^{m}(\{\hat{a}_i\}-\{\hat{b}_i\})$ with $\langle a_1, \cdots, a_m\rangle \simeq \langle b_1, \cdots, b_m\rangle$ [7, Proposition III.2, p.152]. We now wish to apply Lemma 2.1 to $W(R) \simeq \boldsymbol{Z}[G]/K$, with $g=\widehat{-1}$. Certainly $\{\hat{1}\}+\{g\}=\{\hat{1}\}+\{\widehat{-1}\}$ is in M. Furthermore,

$$d_{\widehat{-1}}\Big(\sum_{i=1}^{m}(\{\hat{a}_i\}-\{\hat{b}_i\})\Big) = \Big(\prod_{i=1}^{m}\hat{a}_i\Big)\cdot\Big(\prod_{i=1}^{m}\hat{b}_i\Big)^{-1} = \hat{1}.$$

Thus K is contained in M. Hence Lemma 2.1 applies and $\overline{M}/\overline{M}^2$ is isomorphic to G. The result now follows by composing the explicit isomorphism of Lemma 2.1 with the induced isomorphism of $\overline{M}/\overline{M}^2 \longrightarrow I(R)/I(R)^2$ given by

$$\mathrm{cl}(\textstyle\sum n_i\{g_i\}) \longmapsto (\sum n_i[\langle g_i\rangle]) + I(R)^2.$$

3. The Clifford algebra

Let $Q_2(R)$ be the set of isomorphism classes of ($\boldsymbol{Z}_2$-) graded separable R-algebras which are projective R-modules of rank two. Let L be an abelian group homomorphism from Brauer-Wall group $\mathrm{BW}(R)$ of R onto $Q_2(R)$ given by $L(A)$ $=$class A^{A_0} [11, Theorem 7.10, p.490]. When A is a graded algebra we will write $|A|$ for the algebra considered as an ungraded algebra.

LEMMA 3.1. *Let A be a free rank 4 central separable algebra over a commutative ring R and S a commutative R-algebra. If f is an S-algebra isomorphism of $A\otimes S$ to $M_2(S)$ we will write $N(f,S)$ for the map $A\longrightarrow S$ defined by $N(f,S)(a)=$determinant $(f(a\otimes 1))$.*
Then:

(1) *The image of $N(f,S)$ lies in R and does not depend on the choice of f and S.*

(2) *$N(f,S)$ defines a quadratic form on A.*

Proof. (1) is just [4, Proposition 3.1, p.237], and (2) follows by a routine calculation with 2×2 matrices.

Under the hypotheses of the Lemma, we will write $N:A\longrightarrow R$ for the quadratic form $N(f,S)$, and refer to it as the reduced norm.

COROLLARY 3.2. *Let $[a,b]$ be a non-degenerate quadratic space over R. Then the reduced norm makes $|$Cliff $([a,b])|$ into a quadratic space isometric to $[-a,-b]\perp[1,ab]$. If R is 1-fold full and* Cliff$([a,b])=$Cliff $([c,d])$ *in* BW(R), *it follows that $[-a,-b]\perp[1,ab]\simeq[-c,-d]\perp[1,cd]$.*

Proof. By [6, Lemma 2.1 (iii)], Cliff $([a,b])$ is the rank 4 R-algebra $C=R\oplus Rx\oplus Ry\oplus Rxy$ with $C_0=R\oplus Rxy$, $C_1=Rx\oplus Ry$, $x^2=a$, $y^2=b$, $xy+yx=1$, and C^{C_0} is the rank 2 subalgebra $R\oplus Rxy$. But then, $(xy)^2=x(yx)y=x(1-xy)y=xy-ab$, hence the free rank 2 R-algebra $S=R\oplus Rz$ with $z^2=z-ab$ is Galois [11, Corollary 7.4, p.487].

We now define an R-module homomorphism $f:C\longrightarrow M_2(S)$ by letting

$$f(x)=\begin{bmatrix}0 & a\\ 1 & 0\end{bmatrix},\ f(y)=\begin{bmatrix}0 & z\\ (1-z)/a & 0\end{bmatrix},\ f(1)=1,\text{ and } f(xy)=\begin{bmatrix}1-z & 0\\ 0 & z\end{bmatrix}.$$

Then f is an R-algebra homomorphism just by checking the identities

$$f(xy)=f(x)f(y),\ f(x)^2=\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix},\ f(y)^2=\begin{bmatrix}b & 0\\ 0 & b\end{bmatrix},\text{ and}$$

$$f(x)f(y)+f(y)f(x)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Now let $f^*:C\otimes S\longrightarrow M_2(S)$ be the S-algebra homomorphism induced by f. Since $C\otimes S$ and $M_2(S)$ are free central separable S-algebras of the same rank by [2, Corollary 3.4, p.376], f^* is an isomorphism. Now by directly computing the norm of Lemma 3.1 (1) on the R-basis $\{x, y, 1, xy\}$ we get the quadratic space $[-a,-b]\perp[1,ab]$.

The final conclusion now follows from [9, Lemma 2.3, p.518] and [6, Lemma 1.1] together with the uniqueness in Lemma 3.1.

THEOREM 3.3. *Let (M,B,ϕ) and (N,B',ϕ') be free quadratic spaces of rank 2 over an 1-fold full ring R. Then M is isometric to N if and only if* Cliff $(M)=$Cliff(N) *in* BW(R).

Proof. Clearly, only the sufficiency need be proved. By [6, Lemma 1.1], we may assume $(M, B, \phi)=[a, b]$ and $(N, B', \phi')=[c, d]$.

As noted in the proof of Corollary 3.2 above, $L(\mathrm{Cliff}M)$ is represented by the free rank 2 R-algebra $S=R\oplus Rz$ with $z^2=z-ab$, which is a Galois extension of R with a 2 element Galois group by [11, Corollary 7.4, p.487]. Then, since $(1-z)^2=(1-z)-ab$, S has a unique non-trivial antomorphism j with $j(z)=1-z$. Then defining $\lambda(w)=wj(w)$ for any w in S we define an $S^j=R$ valued quadratic form λ on S. Computing λ on the basis $\{1, z\}$ we see this form is isometric to $[1, ab]$. Therefore, since $L(\mathrm{Cliff}M)=L(\mathrm{Cliff}N)$, we have $[1, ab]\simeq[1, cd]$. Then, by Corollary 3.2 and [10, Theorem 4.3, p.545], $[-a, -b]\simeq[-c, -d]$. It then follows that $M\simeq N$.

COROLLARY 3.4. *Let R be an 1-fold full ring. Then* $\mathrm{Cliff}(I(R)^3)=1$ *in* $\mathrm{BW}(R)$. *If (M, B, ϕ) and (N, B', ϕ') are free quadratic spaces of rank 2 over R with $M=N$ in $W(R)/I(R)^3$, then $M\simeq N$.*

Proof. Since $I(R)$ is generated additively by the forms $\langle 1, a\rangle$, $I(R)^3$ must be generated additively by the forms $\langle 1, a\rangle\langle 1, b\rangle\langle 1, c\rangle$. But

$$\begin{aligned}\mathrm{Cliff}(\langle 1,a\rangle\langle 1,b\rangle\langle 1,c\rangle) &= \mathrm{Cliff}(\langle 1, a, b, ab\rangle)\cdot\mathrm{Cliff}(\langle c, ac, bc, abc\rangle)\\ &= \mathrm{Cliff}(\langle 1, a, b, ab\rangle)\cdot\mathrm{Cliff}(\langle 2\cdot c/2,\\ &\quad 2\cdot ac/2,\ 2\cdot bc/2,\ 2\cdot abc/2\rangle)\\ &= \left[\left(\frac{-a,\ -b}{R}\right)\right]^2\\ &= 1 \text{ in } \mathrm{BW}(R),\end{aligned}$$

by [3, Lemma 3.1] and [9, Lemma 2.9, p.521]. Thus $\mathrm{Cliff}(I(R)^3)=1$ in $\mathrm{BW}(R)$. The last assertion now follows from Theorem 3.3.

4. Main theorem

From now on we will write d for $d_{\hat{1}}$ and any of its induced maps.

THEOREM 4.1. *Let R and S be 2-fold full rings. Then $W(R)/I(R)^3$ is isomorphic to $W(S)/I(S)^3$ if and only if $W(R)$ is isomorphic to $W(S)$.*

Proof. The sufficiency is obvious, since $I(R)$ and $I(S)$ are the unique prime ideals of $W(R)$ and $W(S)$ containing 2 [7, p.156].

Now let f be a ring isomorphism of $W(R)/I(R)^3$ onto $W(S)/I(S)^3$. By the characterization of $I(R)$ and $I(S)$ mentioned above, f induces an isomorphism of $I(R)/I(R)^3$ onto $I(S)/I(S)^3$ and consequently an isomorphism of $I(R)/I(R)^2$ onto $I(S)/I(S)^2$. Then by Theorem 2.2 we get an isomorphism $f': U(R)/U(R)^2\longrightarrow I(R)/I(R)^2\longrightarrow I(S)/I(S)^2\longrightarrow U(S)/U(S)^2$, with

$$\begin{aligned}\hat{a}\longmapsto[\langle 1, -a\rangle]+I(R)^2 &\longmapsto f([\langle 1, -a\rangle]+I(R)^3)+I(S)^2\\ &\longmapsto \mathrm{d}(f([\langle 1, -a\rangle]+I(R)^3)).\end{aligned}$$

Here if we write G_1 for $U(R)/U(R)^2$ and G_2 for $U(S)/U(S)^2$, then f' induces a ring isomorphism of $\boldsymbol{Z}[G_1]\longrightarrow\boldsymbol{Z}[G_2]$. Now, by [7, p.156] this induces a ring homomorphism $f^*: W(R) \longrightarrow W(S)$, if we can show $\langle f'(\hat{1}), f'(\widehat{-1})\rangle \simeq \langle \hat{1}, \widehat{-1}\rangle$ and $\langle f'(\hat{a}), f'(\hat{b})\rangle \simeq \langle f'(\hat{c}), f'(\hat{d})\rangle$ when $\langle a, b\rangle \simeq \langle c, d\rangle$. In fact, this actually proves f^* is an isomorphism, since the same argument applied to $(f')^{-1}$ produces $(f^*)^{-1}$.

Now,

$$\begin{aligned} f'(\widehat{-1}) &= \mathrm{d}(f([\langle 1, 1\rangle] + I(R)^3)) \\ &= \mathrm{d}(f([\langle 1\rangle] + I(R)^3) + f([\langle 1\rangle] + I(R)^3)) \\ &= \mathrm{d}([\langle 1\rangle] + I(S)^3 + [\langle 1\rangle] + I(S)^3) \\ &= \mathrm{d}([\langle 1, 1\rangle] + I(S)^3) \\ &= \widehat{-1}. \end{aligned}$$

Thus the first assertion is proved.

Let x_1 and x_2 be in $I(R)$. Then we may write $f(x_i + I(R)^3) = y_i + I(S)^3$, where each y_i, $i = 1, 2$, is in $I(S)$. If we write $\hat{c}_i = \mathrm{d}(y_i)$, then $\langle 1, -c_i\rangle + I(S)^2 = y_i + I(S)^2$ since both sides have the same image under $\mathrm{d}: I(S)/I(S)^2 \longrightarrow U(S)/U(S)^2$. Therefore we can write $y_i = [\langle 1, -c_i\rangle] + z_i$ for some z_i in $I(S)^2$. Now,

$$\begin{aligned} f(x_1 \cdot x_2 + I(R)^3) &= f(x_1 + I(R)^3) \cdot f(x_2 + I(R)^3) \\ &= (y_1 + I(S)^3) \cdot (y_2 + I(S)^3) \\ &= ([\langle 1, -c_1\rangle] + z_1 + I(S)^3) \cdot ([\langle 1, -c_2\rangle] + z_2 + I(S)^3) \\ &= [\langle 1, -c_1\rangle] \cdot [\langle 1, -c_2\rangle] + I(S)^3. \end{aligned}$$

Now, we substitute $x_1 = [\langle 1, -a\rangle]$ and $x_2 = [\langle 1, -b\rangle]$ into this last formular. Then $f([\langle 1, -a\rangle] \cdot [\langle 1, -b\rangle] + I(R)^3) = [\langle 1, -f'(\hat{a})\rangle] \cdot [\langle 1, -f'(\hat{b})\rangle] + I(S)^3$, since $\hat{c}_1 = f'(\hat{a})$ and $\hat{c}_2 = f'(\hat{b})$ by the definition of f'. But $\langle a, b\rangle \simeq \langle c, d\rangle$ implies

$$\begin{aligned} \langle 1, -a\rangle\langle 1, -b\rangle &= \langle 1, ab\rangle \perp \langle -a, -b\rangle \\ &= \langle 1, cd\rangle \perp \langle -c, -d\rangle \\ &= \langle 1, -c\rangle \perp \langle 1, -d\rangle. \end{aligned}$$

Thus

$$\begin{aligned} &[\langle 1, f'(\hat{a})f'(\hat{b})\rangle] + [\langle -f'(\hat{a}), -f'(\hat{b})\rangle] \\ &\quad \equiv [\langle 1, -f'(\hat{a})\rangle] \cdot [\langle 1, -f'(\hat{b})\rangle] && \bmod I(S)^3 \\ &\quad \equiv [\langle 1, -f'(\hat{c})\rangle] \cdot [\langle 1, -f'(\hat{d})\rangle] && \bmod I(S)^3 \\ &\quad \equiv [\langle 1, f'(\hat{c})f'(\hat{d})\rangle] + [\langle -f'(\hat{c}), -f'(\hat{d})\rangle] && \bmod I(S)^3. \end{aligned}$$

And this congruence becomes $[\langle f'(\hat{a}), f'(\hat{b})\rangle] \equiv [\langle f'(\hat{c}), f'(\hat{d})\rangle] \pmod{I(S)^3}$. Then, by Corollary 3.4, $\langle f'(a), f'(b)\rangle \simeq \langle f'(c), f'(d)\rangle$ which is the required conclusion.

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