Bull. Korean Math. Soc. **22**(1985), No. 2, pp. 79-82

ON CONVERGENCE OF $(S_n-ES_n)/n^{1/r}$, $1<r<2$, FOR PAIRWISE INDEPENDENT RANDOM VARIABLES

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Let $\{X_n\}$ be a sequence of random variables and $S_n=\sum_{i=1}^{n} X_i$. Pyke and Root (1968) proved that if $\{X_n\}$ is a sequence of independent, identically distributed random variables with $E|X_1|^r<\infty$, then $(S_n-ES_n)/n^{1/r}\to 0$ a.s. and in L^r, $1\le r<2$. Chatterji (1969) extended this result to the following form; if $X_1, X_2, \cdots$ are dominated in distribution by a random variable X with $E|X|^r<\infty$ for $1<r<2$, then $(S_n-\sum_{i=1}^{n} E(X_i|X_1, \cdots X_{i-1}))/n^{1/r}\to 0$ a.s. and in L^r. Chow (1971) proved Chatterji's result for L^r-convergence under the weaker hypothesis of uniform integrability of the $|X_n|^r$. For the case of $r=1$, Elton (1981) proved that if $\{X_n\}$ is a martingale difference sequence and identically distributed with $X_1\in L\log L$, then $S_n/n\to 0$ a.s.. Also he showed that if $X\in L^1$ with $E(X)=0$ but $X\notin L\log L$, there exists an identically distributed martingale difference sequence $\{X_n\}$ with X_1 having the same distribution as X such that S_n/n diverges a.s.. Recently Etemadi (1981, 1983) has shown how the classical law of large numbers for independent, identically distributed random variables can be extended in an elementary fashion to the case where the random variables are pairwise independent, identically distributed, i.e., if $\{X_n\}$ is a sequence of pairwise independent, identically distributed random variables with $E|X_1|<\infty$, then $(S_n-ES_n)/n\to 0$ a.s.

In this paper, we study the convergence of normalized partial sums $(S_n-ES_n)/n^{1/r}, 1<r<2$, for pairwise independent random variables (X_n). Namely, we prove that if (X_n) are pairwise independent and are dominated in distribution by a random variable X with $E(|X|^r(\log^+|X|^r)^2)<\infty$, then $(S_n-ES_n)/n^{1/r}\to 0$ a.s.. Under the weaker condition of $E|X|^r<\infty$, we obtain that $(S_n-ES_n)/n^{1/r}\to 0$ in L^1.

We will need the following lemma; see Loève, page 124.

LEMMA 1. *Let the random variables X_n be orthogonal. If $\sum_{n=1}^{\infty} \log^2 n\, E|X_n|^2<\infty$, then $\sum_{n=1}^{\infty} X_n$ converges in L^2 and a.s..*

THEOREM 2. *Let $\{X_i\}$ be a sequence of pairwise independent random variables such that:*

$$(a)\ \sum_{i=1}^{\infty} P(|X_i|\ge i^{1/r})<\infty,$$

(b) $\sum_{i=1}^{n} E(X_i I(|X_i| \geq i^{1/r}))/n^{1/r} \to 0$, and

(c) $\sum_{i=1}^{\infty} \log^2 i \; E(X_i^2 I(|X_i| < i^{1/r}))/i^{2/r} < \infty$, where $1 < r < 2$.

Then $(S_n - ES_n)/n^{1/r} \to 0$ a.s..

Proof. Let $X_i' = X_i I(|X_i| < i^{1/r})$ and $X_i'' = X_i I(|X_i| \geq i^{1/r})$. Then $X_i' + X_i'' = X_i$. Since $\sum_{i=1}^{\infty} P(X_i \neq X_i') = \sum_{i=1}^{\infty} P(|X_i| \geq i^{1/r}) < \infty$, we know by the Borel-Cantelli lemma that $\sum_{i=1}^{n} (X_i - X_i')/n^{1/r} \to 0$ a.s.. Because of pairwise independence of $\{X_i\}$, $\{X_i'\}$ is also pairwise independent, hence $\{(X_i' - E(X_i'))/i^{1/r}\}$ is orthogonal. Now we can apply lemma 1 to the orthogonal sequence $\{(X_i' - E(X_i'))/i^{1/r}\}$ to obtain that $\sum_{i=1}^{\infty} (X_i' - E(X_i'))/i^{1/r}$ converges a.s.. Thus by Kronecker lemma we see that $\sum_{i=1}^{n} (X_i' - E(X_i'))/n^{1/r} \to 0$ a.s.. Therefore $\sum_{i=1}^{n} (X_i - E(X_i')/n^{1/r} \to 0$ a.s..
In the identity

$$\frac{S_n - ES_n}{n^{1/r}} = \frac{\sum_{i=1}^{n} (X_i - E(X_i'))}{n^{1/r}} - \frac{\sum_{i=1}^{n} E(X_i'')}{n^{1/r}},$$

the first term on the right converges to 0 a.s. and the second term converges to 0 by (b). The theorem is thus completely proved.

Let $L(\log L)^2 = \{X \in L^1 : E(|X|(\log^+|X|)^2) < \infty\}$. We need the following lemma to get our main result.

LEMMA 3. *If* $P(|X_i| \geq t) \leq P(|X| \geq t)$ *for all non-negative real numbers* t *and* $|X|^r \in L(\log L)^2$, *then*

$$\sum_{i=1}^{\infty} i^{-2/r} \log^2 i E|X_i'|^2 < \infty \text{ for } 1 \leq r < 2, \text{ where } X_i' = X_i I(|X_i| < i^{1/r}).$$

Proof. First we will estimate $E|X_i'|^2$.

$$\begin{aligned} E|X_i'|^2 &= \int_0^\infty P(|X_i|^2 I(|X_i| < i^{1/r}) \geq t)\,dt = \int_0^{i^{2/r}} (P(t \leq |X_i|^2 < i^{2/r})\,dt \\ &\leq \int_0^{i^{2/r}} P(t \leq |X_i|^2)\,dt \leq \int_0^{i^{2/r}} P(t \leq |X|^2)\,dt = \int_0^{i^{2/r}} P(t \leq |X|^2 < i^{2/r}) \\ &\quad + P(i^{2/r} \leq |X|^2))\,dt = \int_{(|X| < i^{1/r})} |X|^2 dp + i^{2/r} P(i \leq |X|^r). \end{aligned}$$

Hence we obtain that

$$\sum_{i=1}^{\infty} i^{-2/r} \log^2 i E|X_i'|^2 \leq \sum_{i=1}^{\infty} i^{-2/r} \log^2 i \int_{(|X| < i^{1/r})} |X|^2 dp + \sum_{i=1}^{\infty} \log^2 i P(i \leq |X|^r).$$

The proof will be completed by showing that the both terms on the right converge. The convergence of series $\sum_{i=1}^{\infty} \log^2 i P(i \leq |X|^r)$ follows from the following relation.

$$\begin{aligned}\sum_{i=1}^{\infty}\log^2 iP(i\le|X|^r)&=\sum_{i=1}^{\infty}\log^2 i\sum_{j=i}^{\infty}P(j\le|X|^r<j+1)\\&=\sum_{i=1}^{\infty}P(i\le|X|^r<i+1)\sum_{j=1}^{i}\log^2 j\le K\sum_{i=1}^{n}P(i\le|X|^r<i+1)\,i\log^2 i\\&\le KE(|X|^r(\log^+|X|^r)^2)<\infty,\end{aligned}$$

since $\sum_{j=1}^{i}\log^2 j\le\int_2^{i+1}\log^2 x\,dx\le Ki\log^2 i$ for some $K>0$.

It remains to show that

$$\sum_{i=1}^{\infty}i^{-2/r}\log^2 i\int_{\{|X|<i^{1/r}\}}|X|^2dp<\infty.$$

We can show this by the same previous argument as follows.

$$\begin{aligned}\sum_{i=1}^{\infty}i^{-2/r}\log^2 i\int_{\{|X|<i^{1/r}\}}|X|^2dp&=\sum_{i=1}^{\infty}i^{-2/r}\log^2 i\sum_{j=1}^{i}\int_{\{j-1<|X|^r<j\}}|X|^2dp\\&\le\sum_{i=1}^{\infty}i^{-2/r}\log^2 i\sum_{j=1}^{i}j^{2/r}P(j-1\le|X|^r<j)\\&=\sum_{i=1}^{\infty}P(i-1\le|X|^r<i)\,i^{2/r}\sum_{j=i}^{\infty}(j)^{-2/r}\log^2 j\\&\le C\sum_{i=1}^{\infty}P(i-1\le|X|^r<i)\,i\log^2 i\\&\le C\sum_{i=1}^{\infty}P(i\le|X|^r<i+1)\,i\log^2 i\\&\le CE(|X|^r(\log^+|X|^r)^2)<\infty,\end{aligned}$$

Since the second inequality follows from the inequality

$$\sum_{j=i}^{\infty}(j)^{-2/r}\log^2 j\le C\int_i^{\infty}x^{-2/r}\log^2 x\,dx\le Ci^{-2/r+1}\log^2 i$$

where C is an unimportant constant which is allowed to change.

The following lemma is known result[1].

LEMMA 4. *If* $P(|X_i|\ge t)\le P(|X|\ge t)$ *for all non-negative real numbers* t *and* $|X|^r\in L^1$, *then*

$$\sum_{i=1}^{\infty}i^{-1/r}E|X_i''|<\infty\ \textit{for}\ 1<r<2,\ \textit{where}\ X_i''=X_iI(|X_i|\ge i^{1/r}).$$

THEOREM 5. *Let* $\{X_i\}$ *be a sequence of pairwise independent random variables. If* $P(|X_n|\ge t)\le(P|X|\ge t)$ *for all non-negative real numbers* t *and* $|X|^r\in L(\log L)^2$, *then* $(S_n-ES_n)/n^{1/r}\to 0$ a.s. for $1<r<2$.

Proof. Theorem is proved by showing that the conditions of the theorem 2 are satisfied. The condition (a) follows from

$$\sum_{i=1}^{\infty}P(|X_i|^r\ge i)\le\sum_{i=1}^{\infty}P(|X|^r\ge i)\le E|X|^r<\infty.$$

The condition (b) follows from lemma 4 and Kronecker lemma. The condition (c) follows from lemma 3.

COROLLARY 6. *If* $\{X_i\}$ *is a sequence of pairwise independent, identically distributed random variables and* $|X_1|^r\in L(\log L)^2$, *then* $(S_n-nEX_1)/n^{1/r}\to 0$ a.s.

for $1<r<2$.

REMARK 1. When $r=1$, Etemadi (1981, 1983) proved the Corollary under the condition $|X_1|\in L^1$. This Corollary might be true under the weaker condition $|X|^r\in L^1$, but techniques for the cases of $r=1$ or i.i.d. do not seem to work.

REMARK 2. A similar technique to the one used above shows that, under the weaker hypothesis of $|X|^r\in L^1$, $(S_n-ES_n)/n^{1/r}\to 0$ in L^1.

Reference

1. S.D. Chatterji, *An L^p-convergence theorem*, Ann Math. Statist **40**, 1068-1070. (1969)
2. Y.S. Chow, *On the L^p-convergence for $n^{-1/p}S_n$, $0<p<2$*, Ann. Math. Statist. **42**, 393-394. (1971).
3. K.L. Chung, *A Course in Probability Theory*, Academic Press. (1974).
4. S.W. Dharmadhikari, and M. Sreehari, *On convergence in r-mean of normalized partial sums*, Ann. Prob. **3**, 1023-1024. (1975).
5. J. Elton, *A law of large numbers for identically distributed martingale differences*. Ann. Prob. **9**, 405-412. (1981).
6. N. Etemadi, *An elementary proof of the strong law of large numbers*, Z. Wahrsch. Verw, Gebiete **55**, 119-122. (1981).
7. N. Etemadi, *On the laws of large numbers for non-negative random variables*, Jour. Multi. Anal. **13**, 187-193. (1983).
8. M. Loève, *Probability Theory* II, 4th ed. Springer-Verlag. (1977).
9. R. Pyke, and D. Root, *On convergence in r-mean of normalized partial sums*, Ann. Math. Statist. **39**, 379-381. (1968).

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