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SOME GENERATING FUNCTIONS AND HYPERGEOMETRIC TRANSFORMATIONS

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1. For the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by [6, p. 68, Eq. (4.3.2)]

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}, \tag{1.1}$$

the (Jacobi's) generating function

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n=2^{\alpha+\beta}R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta}, \tag{1.2}$$

$$R=(1-2xt+t^2)^{1/2}$$

is well known (cf. [6, p. 69, Eq. (4.4.5)]; see also [4, p. 82, Eq. (1) et seq.]).

An interesting generalization of the generating function (1.2) is the familiar result ([4, p. 145, Eq. (31)]; see also [4, p. 147])

$$\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}P_n^{(\alpha,\beta)}(x)t^n \tag{1.3}$$

$$=F^{p+2:0;0}_{q:1;1}\left[\begin{matrix}\alpha+1,\ \beta+1,\ (a_p):\ \text{—}\ ;\ \text{—}\ ; \\ (b_q):\alpha+1\ ;\ \beta+1\ ;\end{matrix}\ \frac{1}{2}(x-1)t,\ \frac{1}{2}(x+1)t\right],$$

where $(\lambda)_n=\Gamma(\lambda+n)/\Gamma(\lambda)$, $F^{p:r;u}_{q:s;v}$ denotes a general double hypergeometric series defined by [4, p. 63, Eq. (16)]

$$F^{p:r;u}_{q:s;v}\left[\begin{matrix}(a_p):(\alpha_r);(\gamma_u); \\ (b_q):(\beta_s);(\delta_v);\end{matrix}\ x,y\right] \tag{1.4}$$

$$=\sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{m+n}\prod_{j=1}^{r}(\alpha_j)_m\prod_{j=1}^{u}(\gamma_j)_n}{\prod_{j=1}^{q}(b_j)_{m+n}\prod_{j=1}^{s}(\beta_j)_m\prod_{j=1}^{v}(\delta_j)_n}\ \frac{x^m}{m!}\ \frac{y^n}{n!},$$

and (for convenience) (a_p) denotes the array of p parameters $a_1,\cdots,a_p$, with similar interpretations for (b_q), (α_r), (β_s), et cetera.

For $p=q=1$, $a_1=\lambda$, and $b_1=\mu$, the generating function (1.3) assumes the form

$$(1.5)\qquad \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} P_n^{(\alpha,\beta)}(x) t^n$$
$$= F_{1:1;1}^{3:0;0}\left[\begin{matrix}\alpha+1,\ \beta+1,\ \lambda: \text{——};\ \text{——};\\ \mu: \alpha+1;\ \beta+1;\end{matrix}\ \frac{1}{2}(x-1)t,\ \frac{1}{2}(x+1)t\right].$$

Denoting the second member of (1.5) by $\Omega(\lambda, \mu; t)$, and applying the definition (1.4), we have

$$(1.6)\qquad \Omega(\lambda,\mu;t) = \sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}(\beta+1)_{m+n}(\lambda)_{m+n}}{(\mu)_{m+n}(\alpha+1)_m(\beta+1)_n} \frac{\left\{\frac{1}{2}(x-1)t\right\}^m}{m!} \frac{\left\{\frac{1}{2}(x+1)t\right\}^n}{n!},$$

whence (upon setting $m+n=N$, $N \geqq 0$)

$$(1.7)\qquad \Omega(\lambda,\mu;t) = \sum_{N=0}^{\infty} \frac{(\alpha+1)_N(\lambda)_N}{(\mu)_N} \frac{\left\{\frac{1}{2}(x+1)t\right\}^N}{N!}$$
$$\cdot\ {}_2F_1\left[\begin{matrix}-N,\ -\beta-N;\\ \alpha+1;\end{matrix}\ \frac{x-1}{x+1}\right].$$

Making use of Euler's transformation [1, p. 64, Eq. (23)]

$$(1.8)\qquad {}_2F_1\left[\begin{matrix}a,\ b;\\ c;\end{matrix}\ z\right] = (1-z)^{c-a-b}\ {}_2F_1\left[\begin{matrix}c-a,\ c-b;\\ c;\end{matrix}\ z\right],\quad |z|<1,$$

we find from (1.7) that

$$(1.9)\qquad \Omega(\lambda,\mu;t) = \left\{\frac{1}{2}(x+1)\right\}^{-\alpha-\beta-1} \sum_{N=0}^{\infty} \frac{(\alpha+1)_N(\lambda)_N}{(\mu)_N} \frac{\{2t/(x+1)\}^N}{N!}$$
$$\cdot\ {}_2F_1\left[\begin{matrix}\alpha+N+1,\ \alpha+\beta+N+1;\\ \alpha+1;\end{matrix}\ \frac{x-1}{x+1}\right],$$

and (1.5) finally yields the generating function

$$(1.10)\qquad \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} P_n^{(\alpha,\beta)}(x) t^n = \left\{\frac{1}{2}(x+1)\right\}^{-\alpha-\beta-1}$$
$$\cdot\ F_{0:1;2}^{2:0;1}\left[\begin{matrix}\alpha+1,\ \alpha+\beta+1: \text{——};\ \lambda;\\ \text{————}: \alpha+1;\ \mu,\ \alpha+\beta+1;\end{matrix}\ \frac{x-1}{x+1},\ \frac{2t}{x+1}\right].$$

In precisely the same manner we can derive the following alternative form of the generating function (1.3) :

$$(1.11)\qquad \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} P_n^{(\alpha,\beta)}(x) t^n = \left\{\frac{1}{2}(x+1)\right\}^{-\alpha-\beta-1}$$

$$\cdot F^{2:0;p}_{0:1;q+1}\left[\begin{matrix}\alpha+1,\ \alpha+\beta+1:\text{---}; & (a_p); \\ \text{------}:\alpha+1; & (b_q),\ \alpha+\beta+1;\end{matrix}\ \frac{x-1}{x+1},\ \frac{2t}{x+1}\right],$$

which obviously corresponds, when $p=q=1$, to (1.10).

2. Two further special cases of the generating function (1.10) are worthy of note. First of all, in terms of the Appell function F_4 defined by [1, p. 224, Eq. (9)]

$$(2.1)\qquad F_4[\alpha,\beta;\gamma,\delta;x,y]=F^{2:0;0}_{0:1;1}\left[\begin{matrix}\alpha,\beta:-;-; \\ \text{---}:\gamma;\delta;\end{matrix}\ x,\ y\right],$$

the special case of (1.10) when $\lambda=\mu$ becomes

$$(2.2)\qquad \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n$$
$$=\left\{\frac{1}{2}(x+1)\right\}^{-\alpha-\beta-1} F_4\left[\alpha+1,\ \alpha+\beta+1;\alpha+1,\ \alpha+\beta+1;\frac{x-1}{x+1},\ \frac{2t}{x+1}\right],$$

which would naturally yield Jacobi's result (1.2) by virtue of the reduction formula [4, p. 93, Problem 20(iii)]

$$(2.3)\qquad F_4\left[\alpha,\beta;\alpha,\beta;-\frac{x}{(1-x)(1-y)},\ -\frac{y}{(1-x)(1-y)}\right]$$
$$=(1-xy)^{-1}(1-x)^{\beta}(1-y)^{\alpha}.$$

The other interesting special case of the generating function (1.10) occurs when $\lambda=\alpha+\beta+1$, and (with $\mu=\gamma+1$) we thus have

$$(2.4)\qquad \sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n}{(\gamma+1)_n}P_n^{(\alpha,\beta)}(x)\,t^n=\left\{\frac{1}{2}(x+1)\right\}^{-\alpha-\beta-1}$$
$$\cdot F_4\left[\alpha+1,\ \alpha+\beta+1;\alpha+1,\gamma+1;\frac{x-1}{x+1},\ \frac{2t}{x+1}\right],$$

which incidentally follows also from the known result ([2, p. 294, Eq. (45.1.10)]; [4, p. 111, Eq. (31)])

$$(2.5)\qquad \sum_{n=0}^{\infty}\binom{m+n}{n}\frac{(\alpha+\beta+m+1)_n}{(\gamma+1)_n}P_{m+n}^{(\alpha,\beta)}(x)t^n=\binom{m+\alpha}{m}\left\{\frac{1}{2}(x+1)\right\}^{-\alpha-\beta-m-1}$$
$$\cdot F_4\left[\alpha+m+1,\ \alpha+\beta+m+1;\alpha+1,\ \gamma+1;\frac{x-1}{x+1},\ \frac{2t}{x+1}\right],\ m=0,1,2,\cdots$$

upon setting $m=0$.

3. In view of the hypergeometric transformation [5, p. 860, Eq. (10)]

$$(3.1)\qquad F_4[\alpha,\beta;\gamma,\beta;x,y]$$
$$=(1-x-y)^{-\alpha}H_3\left[\alpha,\gamma-\beta;\gamma;\frac{xy}{(x+y-1)^2},\ \frac{x}{x+y-1}\right]$$

or [op. cit., p. 861, Eq. (11)]

(3.2)
$$F_4[\alpha, \beta ; \alpha, \gamma ; x, y] = (1-x-y)^{-\beta} H_3\left[\beta, \gamma-\alpha ; \gamma ; \frac{xy}{(x+y-1)^2}, \frac{y}{x+y-1}\right],$$

the generating function (2.4) [that is, (2.5) with $m=0$] can readily be rewritten in its *equivalent* form (cf. [3, p. 298, Eq. (2.1)])

(3.3)
$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\gamma+1)_n} P_n^{(\alpha,\beta)}(x) t^n = (1-t)^{-\alpha-\beta-1} H_3\left[\alpha+\beta+1, \gamma-\alpha ; \gamma+1 ; \frac{(x-1)t}{2(1-t)^2}, -\frac{t}{1-t}\right],$$

where H_3 denotes one of Horn's double hypergeometric functions defined by [1, p. 225, Eq. (15)]

(3.4)
$$H_3[\alpha, \beta ; \gamma ; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}.$$

For $\gamma=\alpha$, the generating function (3.3) reduces immediately to the familiar result ([2, p. 293, Eq. (45.1.3)]; [4, p. 112, Eq. (34)])

(3.5)
$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(x) t^n = (1-t)^{-\alpha-\beta-1} {}_2F_1\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2) ; \\ \alpha+1 ;\end{matrix} \frac{2(x-1)t}{(1-t)^2}\right].$$

Furthermore, since

(3.6)
$$H_3[\alpha, \beta ; \alpha ; x, y] = \frac{1}{\sqrt{1-4x}}\left[\frac{1+\sqrt{1-4x}}{2}\right]^{1-\alpha}\left[1-\frac{2y}{1+\sqrt{1-4x}}\right]^{-\beta},$$

which follows readily from the definition (3.4) and the known reduction formula [1, p. 101, Eq. (6)]

(3.7)
$${}_2F_1\left[\begin{matrix}\alpha, \alpha+\frac{1}{2} ; \\ 2\alpha ;\end{matrix} z\right] = \frac{1}{\sqrt{1-z}}\left[\frac{1+\sqrt{1-z}}{2}\right]^{1-2\alpha},$$

the special case of (3.3) when $\gamma=\alpha+\beta$ at once yields Jacobi's generating function (1.2).

Finally, by comparing the generating functions (1.5) and (3.3), we obtain the interesting double hypergeometric series transformation

(3.8)
$$F_{1:1;1}^{3:0;0}\left[\begin{matrix}\alpha+1, \beta+1, \alpha+\beta+1 : \text{——} ; \text{——} ; \\ \gamma+1 : \alpha+1 ; \beta+1 ;\end{matrix} xy, (x+1)y\right]$$

$$=(1-y)^{-\alpha-\beta-1}H_3\left[\alpha+\beta+1,\ \gamma-\alpha\ ;\ \gamma+1\ ;\ \frac{xy}{(1-y)^2},\ -\frac{y}{1-y}\right],$$

which is presumably new. In its special case when $\gamma=\alpha$, if we replace α by $\alpha-\beta$ and β by $\beta-1$, (3.8) would yield the reduction formula (cf. [1, p. 238, Eq. (8)] and [4, p. 95, Problem 25(iii)])

$$\text{(3.9)}\qquad F_4[\alpha, \beta\ ;\ \alpha-\beta+1,\ \beta\ ;\ xy, (x+1)y]$$
$$=(1-y)^{-\alpha}\ {}_2F_1\left[\begin{matrix}\frac{1}{2}\alpha,\ \frac{1}{2}\alpha+\frac{1}{2}\ ; \\ \alpha-\beta+1\ ;\end{matrix}\ \frac{4xy}{(1-y)^2}\right]$$

for the Appell function F_4.

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