Bull. Korean Math. Soc. 22(1985), No. 1, pp. 43-52

ON NEW CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z: |z|<1\}$. We denote by $\mathcal{S}$ the subclass of univalent functions $f(z)$ in $\mathcal{A}$ and by $\mathcal{S}^*$ and K the subclasses of $\mathcal{S}$ whose members are starlike with respect to the origin and convex in the unit disk $\mathcal{U}$, respectively. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order $\alpha(0\leqq\alpha<1)$ in the unit disk $\mathcal{U}$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \tag{1.2}$$

fo $z\in\mathcal{U}$. Further, a function $f(z)$ in $\mathcal{A}$ is said to be convex of order $\alpha(0\leqq\alpha<1)$ in the unit disk $\mathcal{U}$ if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \tag{1.3}$$

for $z\in\mathcal{U}$. We denote by $S^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{A}$ whose members satisfy (1.2) and (1.3), respectively. Then, it is well-known that $\mathcal{S}^*(\alpha)\subset\mathcal{S}^*$, $\mathcal{K}(\alpha)\subset\mathcal{K}$ for $0<\alpha<1$ and that $\mathcal{S}^*(0)\equiv\mathcal{S}^*$, $\mathcal{K}(0)\equiv\mathcal{K}$ for $\alpha=0$.

Ruscheweyh [9] introduced the classes $\mathcal{K}_n$ of functions $f(z)$ in $\mathcal{A}$ satisfying

$$\mathrm{Re}\left\{\frac{(z^n f(z))^{(n+1)}}{(z^{n-1}f(z))^{(n)}}\right\}>\frac{n+1}{2} \tag{1.4}$$

for $n\in\mathcal{N}\cup\{0\}$ $(\mathcal{N}=\{1,2,3,\cdots\})$ and $z\in\mathcal{U}$ and showed the basic property

$$\mathcal{K}_{n+1}\subset\mathcal{K}_n \tag{1.5}$$

for each $n\in\mathcal{N}\cup\{0\}$, where $\mathcal{K}_0\equiv\mathcal{S}^*(1/2)$ and $\mathcal{K}_1\equiv\mathcal{K}$.

Let

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$$D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!} \tag{1.6}$$

for $n \in \mathcal{N} \cup \{0\}$. This symbol $D^n f(z)$ was named the nth order Ruscheweyh derivative of $f(z)$ by Al-Amiri [2]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$.

The Hadamard product of two functions $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$ will be denoted by $f*g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.7}$$

then

$$f*g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.8}$$

By using the Hadamard product, Ruscheweyh [9] defined that

$$D^\beta f(z) = \left(\frac{z}{(1-z)^{\beta+1}}\right) * f(z) \qquad (\beta \geqq -1) \tag{1.9}$$

which implies (1.6) for $\beta \in \mathcal{N} \cup \{0\}$. With this notation (1.9), we can observe that the necessary and sufficient condition for a function $f(z)$ in $\mathcal{A}$ to be in the class $\mathcal{K}_n$ is

$$\mathrm{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \frac{1}{2} \qquad (z \in \mathcal{U}). \tag{1.10}$$

Furthermore, in this notation (1.10) also a class $\mathcal{K}_{-1}$ can be defined as the class of functions $f(z)$ in $\mathcal{A}$ satisfying

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2} \qquad (z \in \mathcal{U}). \tag{1.11}$$

Since $\mathcal{K}_0 \equiv S^*(1/2) \subset S^* \subset S$, Ruscheweyh's result implies that $\mathcal{K}_n \subset S^* \subset S$ for each $n \in \mathcal{N} \cup \{0\}$.

Recently, by using the nth order Ruscheweyh derivative of $f(z)$, Singh and Singh [10] introduced the subclass $\mathcal{R}_n$ of $\mathcal{A}$ whose members are characterized by the following condition

$$\mathrm{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \frac{n}{n+1} \qquad (z \in \mathcal{U}) \tag{1.12}$$

for $n \in \mathcal{N} \cup \{0\}$. We can see that $\mathcal{R}_0 \equiv \mathcal{S}^*$ and $\mathcal{R}_n \subset \mathcal{K}_n$ for each $n \in \mathcal{N}$, immediately. Hence $\mathcal{R}_n$ is a subclass of $\mathcal{S}^* \subset \mathcal{S}$ for each $n \in \mathcal{N} \cup \{0\}$. Further we can show that $\mathcal{R}_{n+1} \subset \mathcal{R}_n$ for every $n \in \mathcal{N} \cup \{0\}$.

In the recent years, many classes of functions defined by using the nth order Ruscheweyh derivative of $f(z)$ were studied by Al-Amiri [2], [3], Bulboaca [4], Goel and Sohi [5], [6] and Owa [7], [8].

In the present paper we introduce the following classes $\mathcal{R}_n^*$ by using the n th order Ruscheweyh derivative of $f(z)$.

DEFINITION. We say that $f(z)$ is in the class $\mathcal{R}_n^*$ $(n\in\mathcal{N}\cup\{0\})$, if $f(z)$ defined by

$$f(z)=z-\sum_{k=2}^{\infty} a_k z^k \qquad (a_k\geqq 0) \tag{1.13}$$

satisfies the condition (1.12) for $n\in\mathcal{N}\cup\{0\}$.

2. Distortion theorems

THEOREM 1. *Let the function $f(z)$ be defined by* (1.13). *Then $f(z)$ is in the class $\mathcal{R}_n^*$ if and only if*

$$\sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} a_k \leqq n!. \tag{2.1}$$

Equality holds for the function defined by

$$f(z)=z-\frac{1}{2(n+1)}z^2. \tag{2.2}$$

Proof. Assume that the inequality (2.1) holds and let $|z|=1$. Then we get

$$\begin{aligned}(2.3)\quad &\left|\frac{D^{n+1}f(z)}{D^n f(z)}-1\right| \\ &=\left|\frac{\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots k(k-1)a_k z^{k-1}}{(n+1)!-(n+1)\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots k a_k z^{k-1}}\right| \\ &\leqq \frac{\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots k(k-1)a_k|z|^{k-1}}{(n+1)!-(n+1)\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots k a_k|z|^{k-1}} \\ &\leqq \frac{\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots k(k-1)a_k}{(n+1)!-(n+1)\sum_{k=2}^{\infty}(k+n-1)(k++n-2)\cdots k a_k} \\ &\leqq \frac{1}{n+1}.\end{aligned}$$

This shows that the values of $D^{n+1}f(z)/D^n f(z)$ lie in a circle centered at $w=1$ whose radius is $1/(n+1)$. Thus we can observe that the function $f(z)$ satisfies (1.12) hence further, $f(z)\in\mathcal{R}_n^*$.

For the converse, assume that the function $f(z)$ belongs to the class $\mathcal{R}_n^*$. Then we have

$$\text{(2.4)}\quad \operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} = \operatorname{Re}\left\{\frac{(n+1)!-\sum_{k=2}^{\infty}(k+n)(k+n-1)\cdots k a_k z^{k-1}}{(n+1)!-(n+1)\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots k a_k z^{k-1}}\right\} > \frac{n}{n+1}$$

for $z \in \mathcal{U}$. Choose values of z on the real axis so that $D^{n+1}f(z)/D^n f(z)$ is real. Upon clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we get

$$\text{(2.5)}\quad (n+1)!-\sum_{k=2}^{\infty}(k+n)(k+n-1)\cdots k a_k \geqq \left(\frac{n}{n+1}\right)\left\{(n+1)!-(n+1)\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots k a_k\right\}$$

which gives (2.1).

Further we can see that the function $f(z)$ given by (2.2) is an extreme one for the theorem. Thus we have the theorem.

COROLLARY 1. *Let the function $f(z)$ defined by* (1.13) *be in the class* $\mathcal{R}_n^*$. *Then*

$$\text{(2.6)}\quad a_k \leqq \frac{n!(k-1)!}{(k+n-1)!k}$$

for $k \geqq 2$. *The equality holds for the function $f(z)$ of the form*

$$\text{(2.7)}\quad f(z) = z - \frac{n!(k-1)!}{(k+n-1)!k} z^k.$$

THEOREM 2. *Let the function $f(z)$ defined by* (1.13) *be in the class* $\mathcal{R}_n^*$. Then we have

$$\text{(2.8)}\quad |f(z)| \geqq |z| - \frac{1}{2(n+1)}|z|^2$$

and

$$\text{(2.9)}\quad |f(z)| \leqq |z| + \frac{1}{2(n+1)}|z|^2$$

for $z \in \mathcal{U}$. *The results are sharp.*

Proof. Since $f(z) \in \mathcal{R}_n^*$, in view of Theorem 1, we obtain

$$\text{(2.10)}\quad (n+1)!2\sum_{k=2}^{\infty} a_k \leqq \sum_{k=2}^{\infty}\frac{(k+n-1)!k}{(k-1)!}a_k \leqq n!$$

which implies that

$$\sum_{k=2}^{\infty} a_k \leqq \frac{1}{2(n+1)}. \tag{2.11}$$

Therefore we can show that

$$\begin{aligned} |f(z)| &\geqq |z| - |z|^2 \sum_{k=2}^{\infty} a_k \\ &\geqq |z| - \frac{1}{2(n+1)} |z|^2 \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} |f(z)| &\leqq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \\ &\leqq |z| + \frac{1}{2(n+1)} |z|^2 \end{aligned} \tag{2.13}$$

for $z \in \mathcal{U}$.

Finally, by taking the function

$$f(z) = z - \frac{1}{2(n+1)} z^2, \tag{2.14}$$

we can see that the results of the theorem are sharp.

COROLLARY 2. *Let the function $f(z)$ defined by (1.13) be in the class $\mathcal{R}_n^*$. Then $f(z)$ is included in a disk with its center at the origin and radius r given by*

$$r = \frac{2n+3}{2(n+1)}. \tag{2.15}$$

THEOREM 3. *Let the function $f(z)$ defined by (1.13) be in the class $\mathcal{R}_n^*$. Then we have*

$$|f'(z)| \geqq 1 - \left(\frac{1}{n+1}\right) |z| \tag{2.16}$$

and

$$|f'(z)| \leqq 1 + \left(\frac{1}{n+1}\right) |z| \tag{2.17}$$

for $z \in \mathcal{U}$. The results are sharp.

Proof. In view of Theorem 1, we obtain

$$\begin{aligned} (n+1)! \sum_{k=2}^{\infty} k a_k &\leqq \sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} a_k \\ &\leqq n! \end{aligned} \tag{2.18}$$

which implies that

$$\sum_{k=2}^{\infty} ka_k \leqq \frac{1}{n+1}. \tag{2.19}$$

Hence, with the aid of (2.19), we have

$$\begin{aligned} |f'(z)| &\geqq 1-|z| \sum_{k=2}^{\infty} ka_k \\ &\geqq 1-\left(\frac{1}{n+1}\right)|z| \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} |f'(z)| &\leqq 1+|z| \sum_{k=2}^{\infty} ka_k \\ &\leqq 1+\left(\frac{1}{n+1}\right)|z| \end{aligned} \tag{2.21}$$

for $z\in\mathcal{U}$. Further the results of the theorem are sharp for the function $f(z)$ given by (2.14).

COROLLARY 3. *Let the function $f(z)$ defined by* (1.13) *be in the class $\mathcal{R}_n{}^*$. Then $f'(z)$ is included in a disk with its center at the origin and radius R given by*

$$R=\frac{n+2}{n+1}. \tag{2.22}$$

THEOREM 4. *Let the function $f(z)$ defined by* (1.13) *be in the class $\mathcal{R}_n{}^*$. Then we have*

$$|f''(z)|<\frac{2}{n+1} \tag{2.23}$$

for $z\in\mathcal{U}$.

Proof. With the aid of Theorem 1, we can see that

$$\begin{aligned} \frac{(n+1)!}{2} \sum_{k=2}^{\infty} k(k-1)a_k &\leqq \sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} a_k \\ &\leqq n! \end{aligned} \tag{2.24}$$

which gives that

$$\sum_{k=2}^{\infty} k(k-1)a_k<\frac{2}{n+1}. \tag{2.25}$$

Consequently we have

$$\begin{aligned} |f''(z)| &\leqq \sum_{k=2}^{\infty} k(k-1)a_k|z|^{k-2} \\ &<\frac{2}{n+1}. \end{aligned} \tag{2.26}$$

COROLLARY 4. *Let the function* $f(z)$ *defined by* (1.13) *be in the class* $\mathcal{R}_n^*$. *Then* $f''(z)$ *is included in a disk with its center at the origin and radius* $2/(n+1)$.

3. Closure theorems

THEOREM 5. *Let the functions*

$$f_i(z)=z-\sum_{k=2}^{\infty} a_{k,i}z^k \qquad (a_{k,i}\geqq 0) \tag{3.1}$$

be in the class $\mathcal{R}_n^*$ *for every* $i=1,2,3,\cdots,m$. *Then the function* $h(z)$ *defined by*

$$h(z)=\sum_{i=1}^{m} c_i f_i(z) \qquad (c_i\geqq 0) \tag{3.2}$$

is also in the same class $\mathcal{R}_n^*$, *where*

$$\sum_{i=1}^{m} c_i=1. \tag{3.3}$$

Proof. By means of the definition of $h(z)$, we obtain

$$h(z)=z-\sum_{k=2}^{\infty}\left(\sum_{i=1}^{m} c_i a_{k,i}\right) z^k. \tag{3.4}$$

Further, since $f_i(z)$ are in $\mathcal{R}_n^*$ for every $i=1,2,3,\cdots,m$, we get

$$\sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} a_{k,i}\leqq n! \tag{3.5}$$

for every $i=1,2,3,\cdots,m$. Hence we can see that

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!}\left(\sum_{i=1}^{m} c_i a_{k,i}\right) &= \sum_{i=1}^{m} c_i\left\{\sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} a_{k,i}\right\} \\ &\leqq \left(\sum_{i=1}^{m} c_i\right)n! \\ &= n! \end{aligned} \tag{3.6}$$

with the aid of (3.5). This proves that the function $h(z)$ is in the class $\mathcal{R}_n^*$ by means of Theorem 1. Thus we have the theorem.

THEOREM 6. *Let*

$$f_1(z)=z \tag{3.7}$$

and

$$f_k(z)=z-\frac{(k-1)!n!}{(k+n-1)!k} z^k \tag{3.8}$$

for $k\in\mathcal{N}-\{1\}$. *Then* $f(z)$ *is in the class* $\mathcal{R}_n{}^*$ *if and only if it can be expressed in the form*

(3.9) $$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z),$$

where $\lambda_k\geqq 0$ and

(3.10) $$\sum_{k=1}^{\infty}\lambda_k=1$$

Proof. Assume that

(3.11) $$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z)$$
$$=z-\sum_{k=2}^{\infty}\frac{(k-1)!n!}{(k+n-1)!k}\lambda_k z^k.$$

Then we have

(3.12) $$\sum_{k=2}^{\infty}\left\{\frac{(k+n-1)!k}{(k-1)!}\quad\frac{(k-1)!n!}{(k+n-1)!k}\lambda_k\right\}\leqq n!.$$

Consequently we can see that $f(z)$ is in the class $\mathcal{R}_n{}^*$ by means of Theorem 1.

Conversely, suppose that $f(z)$ is in the class $\mathcal{R}_n{}^*$. Again, with the aid of Theorem 1, we get

(3.13) $$a_k\leqq\frac{(k-1)!n!}{(k+n-1)!k}$$

for $k\in\mathcal{N}-\{1\}$. Setting

(3.14) $$\lambda_k=\frac{(k+n-1)!k}{(k-1)!n!}a_k$$

for $k\in\mathcal{N}-\{1\}$ and

(3.15) $$\lambda_1=1-\sum_{k=2}^{\infty}\lambda_k,$$

we obtain the representation (3.9). This completes the proof of the theorem.

4. Modified Hadamard product

Let $f(z)$ be defined by (1.13) and $g(z)$ be defined by

(4.1) $$g(z)=z-\sum_{k=2}^{\infty}b_k z^k \qquad (b_k\geqq 0).$$

And let $f*g(z)$ denote the modified Hadamard product of $f(z)$ and $g(z)$, that is,

(4.2) $$f*g(z)=z-\sum_{k=2}^{\infty}a_k b_k z^k.$$

THEOREM 7. *Let the functions $f_i(z)$ defined by* (3.1) *be in the class $\mathcal{R}_{n_i}$ for each $i=1, 2, 3, \cdots, m$, respectively. Then the modified Hadamard product $f_1*f_2*\cdots*f_m(z)$ belongs to the class $\mathcal{R}_n^*$, where $n=\min_{1\leq i\leq m}\{n_i\}$.*

Proof. Since $f_i(z)\in\mathcal{R}_{n_i}^*$ for each $i=1, 2, 3, \cdots, m$, we have

$$a_{k,i}\leqq\frac{1}{2(n_i+1)} \qquad (i=1, 2, 3\cdots, m) \tag{4.3}$$

by using Theorem 1. Therefore we can see that

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{(k+n-1)!k}{(k-1)!}\left(\prod_{i=1}^{n}a_{k,i}\right)&\leqq\frac{n!2(n+1)}{\prod_{i=1}^{m}2(n_i+1)}\\&\leqq n!\end{aligned} \tag{4.4}$$

with the aid of (4.3), where $n=\min_{1\leq i\leq m}\{n_i\}$. This shows that the modified Hadamard product $f_1*f_2*\cdots*f_m(z)$ is in the class $\mathcal{R}_n^*$.

COROLLARY 5. *Let the functions $f_i(z)$ defined by* (3.1) *be in the same class $\mathcal{R}_n^*$ for every $i=1, 2, 3, \cdots, m$. Then the modified Hadamard product $f_1*f_2*\cdots*f_m(z)$ also is in the class $\mathcal{R}_n^*$.*

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