Bull. Korean Math. Soc, **22** (1985), No. 1, pp. 23-29

SPECTRA OF THE IMAGES UNDER THE FAITHFUL *-REPRESENTATION OF $L(H)$ ON K

HYUNG KOO CHA

1. Introduction

Let H be an arbitrary complex Hilbert space. We constructed an extension K of H by means of weakly convergent sequences in H and the Banach limit.

Let ϕ be the faithful *-representation of $L(H)$ on K. In this note, we investigated the relations between spectra of T in $L(H)$ and $\phi(T)$ in $L(K)$ and we obtained the following results:

1) If T is a compact operator on H, then $\phi(T)$ is also a compact operator on K (Proposition 6),
2) $\sigma_e(\phi(T))\subset\sigma_e(T)$ for any operator $T\in L(H)$ (Corollary 10),
3) For every operator $T\in L(H)$, $\sigma_{ap}(\phi(T))=\sigma_{ap}(T)=\sigma_p(\phi(T))$ (Lemma 12, 13) and $\sigma_c(\phi(T))=\phi$ (Theorem 15).

2. Notations and terminology

Throughout this paper, we used the following notations and terminology. The *-algebra of all bounded linear operators on H is denoted $L(H)$.

The spectrum, the point spectrum, the compression spectrum, the approximate point spectrum and the continuous spectrum of an operator T are denoted $\sigma(T)$, $\sigma_p(T)$, $\sigma_{com}(T)$, $\sigma_{ap}(T)$ and $\sigma_c(T)$, respectively. For those spectra of T, we have the following results ([1], [2]):

a) $\sigma_p(T)=\{\lambda\in\boldsymbol{C}\mid T-\lambda I \text{ is a left divisor of zero in } L(H)\}$,
b) $\sigma_{com}(T)=\{\lambda\in\boldsymbol{C}\mid T-\lambda I \text{ is a right divisor of zero in } L(H)\}$,
c) $\sigma_{ap}(T)=\{\lambda\in\boldsymbol{C}\mid T-\lambda I \text{ is not bounded below}\}$
$=\{\lambda\in\boldsymbol{C}\mid \not\exists\varepsilon>0 \text{ such that } (T-\lambda I)^*(T-\lambda I)\geq\varepsilon I\}$.

We denoted by LIM the Banach limit defined for bounded sequences $\{x_n\}$ of complex numbers, which satisfies the following properties,

a) LIM $x_n=$LIM x_{n+1}
b) LIM $(\alpha x_n+\beta y_n)=\alpha$LIM $x_n+\beta$LIM y_n, for all $\alpha,\beta\in\boldsymbol{C}$,
c) LIM $x_n\geq0$ when $x_n\geq0$ for all n,
d) LIM $x_n=\lim x_n$ whenever $\{x_n\}$ is convergent,

e) $\liminf x_n \leq \text{LIM}\, x_n \leq \limsup x_n$ when $\{x_n\}$ is real sequence,
f) $\text{LIM}(x_n{}^*) = (\text{LIM}\ x_n)^*$ ([1], [5], [7])

3. Construction of K

Let $\mathscr{B}$ be the set of all weakly convergent sequences $s=\{x_n\}$ in H. If $s=\{x_n\}$ and $t=\{y_n\}$, we write $s=t$ in case $x_n=y_n$ for all n. The set $\mathscr{B}$ is clearly a vector space for the operations

$$s+t=\{x_n+y_n\} \text{ and } \lambda s=\{\lambda x_n\} \text{ for all } \lambda\in \boldsymbol{C}.$$

Let $s=\{x_n\}$ and $t=\{y_n\}$ be the elements of $\mathscr{B}$. In a Hilbert space, since every weakly convergent sequence is bounded in H ([9]), it is permissible to define

$$\psi(s,t)=\text{LIM}\ \ (x_n|y_n).$$

Evidently, ψ is a positive symmetric sesquilinear form on $\mathscr{B}$ and $|\psi(s,t)|^2\leq\psi(s,s)\ \psi(t,t)$ ([2]).

Let $\mathscr{N}=\{s|\psi(s,s)=0\}=\{s|\psi(s,t)=0 \text{ for all } t\in\mathscr{B}\}$.
Then $\mathscr{N}$ is clearly a linear subspace of $\mathscr{B}$. Thus, the quotient vector space $\mathscr{D}=\mathscr{B}/\mathscr{N}$ becomes an inner product space on definining

$$(s'|t')=\psi(s,t),$$

where s' and t' are the cosets $s+\mathscr{N}$ and $t+\mathscr{N}$, respectively.

If x is in H, we write $\{x\}$ for the sequence all of whose terms are x and x' for the coset $\{x\}+\mathscr{N}$. Obviously, $(x'|y')=(x|y)$ and $x\longrightarrow x'$ is an isometric linear mapping of H onto a closed linear subspace K' of $\mathscr{D}$.

Let K be the completion of $\mathscr{D}$. Thus, $K'\subset\mathscr{D}\subset K$ and $\mathscr{D}$ is a dense linear subspace of K. In this case, $\mathscr{B}$ is invariant under T for each $T\in L(H)$. For, let s be an arbitrary element in $\mathscr{B}$ and let $s=\{x_n\}$.

By the consruction of $\mathscr{B}$, $(x_n|y)\longrightarrow(x|y)$ for all $y\in H$, whence $(Tx_n|y)=(x_n|T^*y)\longrightarrow(x|T^*y)=(Tx|y)$. Therefore, $\{Tx_n\}$ is the weakly convergent sequence.

4. A *-represention $L(H)$ on K

In the remark of the preceding section, since we knew that if $s\in\mathscr{B}$, then $Ts\in\mathscr{B}$ for each operator $T\in L(H)$, we can determines an operator $\phi(T)\in L(K)$ for any $T\in L(H)$ as follows.

Defining $T_0s=\{Tx_n\}$ for each $s=\{x_n\}\in\mathscr{B}$, we have a linear mapping $T_0:\mathscr{B}\longrightarrow\mathscr{B}$ such that $\psi(T_0s,T_0s)\leq\|T\|^2\psi(s,s)$. In particular, if s is in $\mathscr{N}$, then T_0s is also in $\mathscr{N}$. It follows that $\{x_n\}'\longrightarrow\{Tx_n\}'$ is a well-defined linear mapping of $\mathscr{D}$ into $\mathscr{D}$, which we denote $\phi(T)$. Thus, $\phi(T)s'=(T_0s)'$ and the inequality $(\phi(T)u|\phi(T)u)\leq\|T\|^2(u|u)$ for each u in $\mathscr{D}$ shows that $\phi(T)$ is continuous. In particular, since $\phi(T)\{x\}'=(Tx)'$ for any x in H, it is clear that $\|\phi(T)\|\geq\|T\|$.

Thus, we have $\|\phi(T)\|=\|T\|$.

5. Main results

PROPOSITION 1. *The mapping $\phi: L(H)\longrightarrow L(K)$ is a faithful *-representation of $L(H)$ on K such that $\phi(I)$ is the identity on K.*

Proof. Let $s=\{x_n\}\in\mathscr{B}$ and let $s'=s+\mathscr{N}$. Then

$$\begin{aligned}\phi(S+T)s' &= ((S+T)_0 s)' = \{(S+T)x_n\}' \\ &= \{Sx_n+Tx_n\}' = \{Sx_n\}'+\{Tx_n\}' \\ &= (S_0s)'+(T_0s)' = \phi(S)s'+\phi(T)s' \\ &= (\phi(S)+\phi(T))s'\end{aligned}$$

Hence, we have $\phi(S+T)=\phi(S)+\phi(T)$.

Let $s=\{x_n\}\in\mathscr{B}$ and $s'=s+\mathscr{N}$. Then

$$\begin{aligned}\phi(ST)s' &= ((ST)_0 s)' = \{STx_n\}' = \{S(Tx_n)\}' \\ &= S_0\{Tx_n\})' = \phi(S)\{Tx_n\}' \\ &= \phi(S)(T_0s)' \\ &= \phi(S)\phi(T)s'.\end{aligned}$$

Hence, we have $\phi(ST)=\phi(S)\phi(T)$.

For all $s'=\{x_n\}+\mathscr{N}$ and $t'=\{Y_n\}+\mathscr{N}$ belong to $\mathscr{D}$,

$$\begin{aligned}(s'\,|\,\phi(T)^*t') &= (\phi(T)s'\,|\,t') \\ &= \mathrm{LIM}(Tx_n\,|\,y_n) \\ &= \mathrm{LIM}(x_n\,|\,T^*y_n) \\ &= (s'\,|\,\phi(T^*)y_n)\end{aligned}$$

Since s' and t' are arbitrary, we have $\phi(T^*)=\phi(T)^*$.

And we have $\phi(I)$ is the identity on K, for, $\phi(I)s'=(I_0s)'=\{Ix_n\}'=\{x_n\}'=s'$ for each $s'\in\mathscr{D}$. Obviously, ϕ is injective.

It follows from the above results that ϕ is a faithful *-representation of $L(H)$ on K with identity $\phi(I)$.

PROPOSITION 2. *For each operators $T\in L(H)$,*

(a) *If T is invertible on H, then $\phi(T)$ is invertible on K and $\phi(T^{-1})=\phi(T)^{-1}$.*

(b) *For any weakly convergent sequence $\{x_n\}$ in H, there exists a vector $u\in K$ such that*

$$\|\phi(T)u\|^2=LIM\|Tx_n\|^2, \text{ for all } T\in L(H).$$

Proof. Suppose that T is invertible on H. Then

$$s'=\{TT^{-1}x_n\}'=\phi(T)\phi(T^{-1})s' \text{ and}$$

$$s' = \{T^{-1}Tx_n\}' = (T^{-1})\phi(T)s' \text{ for each } s' \in \mathcal{D}.$$

Thus, we have $\phi(I) = \phi(T)\phi(T^{-1}) = \phi(T^{-1})\phi(T)$. By the uniqueness of inverse, $\phi(T^{-1}) = \phi(T)^{-1}$.
The second is clear from the definition.

From Proposition 2, we have the following result.

CORPOLLARY 3. *$\sigma(\phi(T)) = \sigma(T)$ for all operators $T \in L(H)$.*

PRPOOSITION 4. *If M is a spectral set for T, then M is also a spectral set for $\phi(T)$.*

Proof. Suppose that M is a spectral set for T. Then by the definition of spectral set for T, $\sigma(T) \subset M$ and $\|f(T)\| \leq \|f\|_M = \sup\{|f(\lambda)| : \lambda \in M\}$ for all $f \in \boldsymbol{C}(t\,;M)$ ([2]).
Since $\phi(f(T)) = f(\phi(T))$ for all $f \in \boldsymbol{C}(t\,;M)$,

$$\begin{aligned} \|f(\phi(T))\| &= \|\phi(f(T))\| \\ &\leq \|\phi\|\ \|f(T)\| \\ &= \|f(T)\| \\ &< \|f\|_M. \end{aligned}$$

Therefore, M is a spectral set for $\phi(T)$.

PROPOSITION 5. *$W(T)^- = W(\phi(T))^-$, where $W(T)^-$ is the closure of the numerical range of T.*

Proof. For each element $\alpha \in W(T)^-$, there exists a sequence $\{x_n\}$ of vectors x_n in H with $\|x_n\| = 1$ such that $(Tx_n | x_n) \longrightarrow \alpha$ if and only if $(\phi(T)x_n' | x_n') \longrightarrow \alpha$, since $\|x_n\| = \|x_n'\|$ and $(Tx_n | x_n) = (\phi(T)x_n' | x_n')$, where $x_n' = \{x_n\} + \mathcal{N}$ for all n. Therefore, $\alpha \in W(\phi(T))^-$.

PROPOSITION 6. *If T is a compact operator on H, then $\phi(T)$ is also a compact operator on K.*

Proof. Let $s_n' = s_n + \mathcal{N}$ be a bounded sequence in K with $s_n = \{x_{m,n}\}$. Then there exists $x_n \in H$ such that x_n is the weak limit of the sequence $\{x_{m,n}\}$.
Since $\|x_n\| \leq \liminf_m \|x_{m,n}\|$ ([9]), the sequence $\{x_n\}$ is a bounded sequence. Thus, there is a weakly convergent subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ and since T is the compact operator, we have $\|Tx_{m,n} - Tx_n\| \longrightarrow 0$ as $m \longrightarrow \infty$.
Let $\{x_{m,n_i}\}$ be the sequence which have the weak limit x_{n_i} and let $s = \{x_{n_i}\}$. Then,

$$\begin{aligned} \|\phi(T)(s_{n_i})' - \phi(T)s'\|^2 &= \|\phi(T)(s_{n_i}' - s')\|^2 \\ &= \|\phi(T)\{x_{m,n_i} - x_{n_i}\}'\|^2 \end{aligned}$$

$$= \mathrm{LIM}\|T(x_{m,n_i}-x_{n_i})\|^2$$
$$= \mathrm{LIM}\|Tx_{m,n_i}-Tx_{n_i}\|^2$$
$$= 0.$$

Hence $\phi(T)$ is a compact operator on K.

The set of all compact operators on H is denoted $K(H)$. Let π be the natural quotient map of $L(H)$ onto the Calkin algebra $L(H)/K(H)$. The essentially normal operator is the operator $T\in L(H)$ such that $\pi(T)$ is normal and the essentially unitary operator is the operator $T\in L(H)$ such that $\pi(T)$ is unitary.

COROLLARY 7. *If T is an essentially normal operator on H, then $\phi(T)$ is an essentially normal operator on K.*

Proof. Suppose T is an essentially normal operator. Then T^*T-TT^* is the compact operator and so by Proposition 6, $\phi(T^*)\phi(T)-\phi(T)\phi(T^*)$ is the compact operator in K. Hence $\phi(T)$ is an essentially normal operator in K.

COROLLARY 8. *If T is an essentially unitary operator on H, then $\phi(T)$ is an essentially unitary operator on K.*

Proof. Suppose T is an essentially unitary operator. Then T^*T-I is the compact operator and by Proposition 6, $\phi(T^*)\phi(T)-\phi(I)$ is the compact operator on K. Hence $\phi(T)$ is an essentially unitary operator on K.

PROPOSITION 9. *If T is a Fredholm operator on K, then $\phi(T)$ is a Fredholm operator on K.*

Proof. Suppose that T is a Fredholm operator. Then since $\pi(T)$ is an invertible element in $L(H)/K(H)$, there exists an operator $S\in L(H)$ such that.

$$TS=I+K_1 \text{ and } ST=I+K_2$$

where K_1 and K_2 are elements in $K(H)$. By Propositions 1 and 6,

$$\phi(T)\phi(S)=\phi(I)+\phi(K_1) \text{ and } \phi(S)\phi(T)=\phi(I)+\phi(K_2)$$

Hence $\pi\phi(T)$ is an invertible element in $L(K)/K(K)$.

We have the following immediate consequence:

COROLLARY 10. *$\sigma_e(\phi(T))\subset\sigma_e(T)$ for any operator $T\in L(H)$, where $\sigma_e(T)$ is the essentially spectrum of T and $\sigma_e(T)=\sigma(\pi(T))$.*

Suppose $T\geq 0$, that is, $(Tx|x)\geq 0$ for all x in H. If $u=\{x_n\}'$ is an element in $\mathcal{D}$, then $(Tx_n|x_n)\geq 0$ for all n. Thus,

$$(\phi(T)u|u)=\mathrm{LIM}(Tx_n|x_n)\geq 0$$

Therefore,

$$(\phi(T)v|v)\geq 0 \text{ for all } v \text{ in } K.$$

By the above result, we have the following result:

PROPOSITION 11. *The operator T is positive if and only if $\phi(T)$ is positive.*

LEMMA 12. *If T is an operator on H, then $\sigma_{ap}(\phi(T))=\sigma_{ap}(T)$.*

Proof. A complex number λ does not belong to $\sigma_{ap}(T)$ if and only if there exists a positive number ε such that $(T-\lambda I)^*(T-\lambda I)\geq\varepsilon I$ if and only if $(\phi(T)-\lambda\phi(I))^*(\phi(T)-\lambda\phi(I))\geq\varepsilon\phi(I)$. So we have the desired equality.

LEMMA 13. *If T is an operator in $L(H)$, then*

$$\sigma_{ap}(T)=\sigma_p(\phi(T)).$$

Proof. The relations $\sigma_p(\phi(T))\subset\sigma_{ap}(\phi(T))=\sigma_{ap}(T)$ have already been known.

Conversely, a complex number λ belong to $\sigma_{ap}(T)$ if and only if there exists a sequence $\{x_n\}$ of vectors x_n in H with $\|x_n\|=1$ such that $\|(T-\lambda I)x_n\|\longrightarrow 0$. Since $\{x_n\}$ is the bounded sequence, there exists a weakly convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Let u be the subseqence $\{x_{n_i}\}$ of $\{x_n\}$. Then $u'=u+\mathscr{U}\in K$ and $\|u'\|=1$. Thus,

$$\begin{aligned}\|\phi(T)u'-\lambda\phi(I)u'\|^2&=((\phi(T)-\lambda\phi(I))u'\,|\,(\phi(T)-\lambda\phi(I))u')\\&=(((T-\lambda I)_0u)'\,|\,((T-\lambda I)_0u)')\\&=(\{(T-\lambda I)x_{n_i}\}'\,|\,\{(T-\lambda I)x_{n_i}\}')\\&=\mathrm{LIM}((T-\lambda I)x_{n_i}\,|\,(T-\lambda I)x_{n_i})\\&=\mathrm{LIM}\|(T-\lambda I)x_{n_i}\|\\&=0\end{aligned}$$

Hence we have $\phi(T)u'=\lambda u'$ and so $\lambda\in\sigma_p(\phi(T))$.

LEMMA 14. *$\sigma_{com}(T)\subset\sigma_{com}(\phi(T))$ for any operator $T\in L(H)$.*

Proof. Suppose that $\lambda\in\sigma_{com}(T)$. Since $T-\lambda I$ is a right divisor of zero in $L(H)$, there exists an operator $S\in L(H)$ such that $S(T-\lambda I)=0$. Thus,

$$\phi(S)(\phi(T)-\lambda\phi(I))=\phi(S(T-\lambda I))=0.$$

Therefore, we have $\lambda\in\sigma_{com}(\phi(T))$.

THEOREM 15. *For every operator $T\in L(H)$, $\sigma_c(\phi(T))=\phi$.*

Proof. $$\begin{aligned}\sigma_c(\phi(T))&=\sigma(\phi(T))-(\sigma_p(\phi(T))\cup\sigma_{com}(\phi(T)))\\&\subset\sigma(T)-(\sigma_{ap}(T)\cup\sigma_{com}(T))=\phi.\end{aligned}$$

References

1. S.K. Berberian, *Approximate proper vectors,* Proc. Amer. Math. Soc. **13** (1962), 111-114.
2. S.K. Berberian, *Lectures in functional analysis and operator theory,* Springer-Verlag, N.Y., 1974.
3. Ronald, G. Douglas, *Banach algebra techniques in operator theory,* Academic Press, N.Y. and London, 1972.
4. H.R. Dowson, *Spectral theory of linear operators,* Academic Press Inc., London, 1978.
5. N. Dunffrd, J.T. Schwarz, *Linear operator I,* Interscience Pub. Inc., N.Y., 1964.
6. R. Lange, *Sets essentially unitary operators,* Trans. Amer. Math. Soc. **281** (1984) 65-75.
7. R. Larsen, *Functional analysis,* Marcel Dekker Inc., N.Y., 1973.
8. J.G. Stampfli, *Compact perturbations, normal eigenvalues and a problem salinas,* J. London Math. Soc. (2) **9**(1974) 165-175.
9. J. Weidmann, *Linear operators in Hilbert spaces,* Springer-Verlag, N.Y., 1980.

Hanyang University
Seoul 133, Korea