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MODULES OVER THE φ- DIFFERENTIAL POLYNOMIAL RINGS

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I. Introduction

The differential polynomial ring $A[X, D]$ has been studied by many authors J. Cozzens, C. Faith, R. E. Johnson and D. Mathis and others. The main purpose of the present paper is to study some properties of φ-differential polynomial ring $A[X, D, \varphi]$ and modules over the φ-differential polynomial ring $A[X, D, \varphi]$.

II. Preliminaries

Throughout the paper, A will be donoted an associative ring with an identity element, and let $\varphi : A \to A$ be a ring monomorphism. Then a φ-derivation of A is a mapping $D : A \to A$ such that $D(a+b) = D(a) + D(b)$, and $D(ab) = \varphi(a)D(b) + D(a)b$ for all a, b in A. If φ is an identity map, then D is called a derivation of A. For non-commutative rings, for each element a not in the center of the ring A, an inner derivation D_a is defined by $D_a(y) = ya - ay$, for all y in A. A derivation is outer if it is not inner.

The existence of such φ-derivation can be seen following example ([3]). Let K be a field contained in A such that $\dim_K A = 2$. Then we can choose a basis 1 and x in A, where 1 is the identity element in A and x is not contained in K. Thus we can represent, for each a in A, $xa = D(a)1 + \varphi(a)x$ and this defines a ring monic of A and a φ-derivation D.

DEFINITION 1. Let A be a ring and let M be a right A-module. Consider a polynomial ring $A[X, D, \varphi]$ which is defined by usual polynomial addition and multiplication defined by $aX = X\varphi(a) + D(a)$ where $\varphi : A \to A$ is a ring monomorphism and D is a φ-derivation on A, such polynomial ring $A[X, D, \varphi]$ is called a φ-differential polynomial ring in D.
For such a ring derivation $D : A \to A$, we define a structure map $D' : M \to M$ such that

$$D'(m+n) = D'(m) + D'(n)$$
$$D'(ma) = D'(m)\varphi(a) + mD(a)$$

Such D' is an element of $\mathrm{Hom}_Z(M, M)$ and it is called (D, φ)-structure map on right A-module M, and $\mathrm{St}(M)$ denotes the collection of all (D, φ)-structure maps on M. If φ is an identity map on A, we call φ-differential polynomial ring as differential polynomial ring and (D,φ)-structure map is called D-structure map ([5]).

REMARK. The extistence of such (D,φ)-structure maps can be proved, if the module is a projective module. ([5]).

DEFINITION 2. Let M and N be right A-modules and let D' and G' be (D,φ)-structure maps on M and N, respectively. If there is a map $f: M \to N$ such that $G'f = fD'$, we call such map f as φ-maps on M into N. If φ is an identity map on A, such φ-map f will be called D-map as in the [5].

III. Right $A[X, D, \varphi]$-modules

PROPOSITION 1. *A right A-module can be extended to $A[X,D,\varphi]$-module if and only if there exist at least one (D,φ)-structure map on M.*

Proof. Let M be a right $A[X,D,\varphi]$-module, then we have $(ma)X = m(aX)$, where m in M and a in A. Using this we can construct a (D,φ)-structure map on M, by defining $D'(m) = mX$ for all m in M.

Since
$$\begin{aligned} D'(ma) &= (ma)X = m(aX) = m(X\varphi(a) + D(a)) \\ &= (mX)\varphi(a) + mD(a) \\ &= D'(m)\varphi(a) + mD(a). \end{aligned}$$

Conversely, if there exists a (D,φ)-structure map D' on M, we define $D'(m) = mX$. Then M is a right $A[X,D,\varphi]$-module.

PROPOSITION 2. *If a right A-module M has a (D,φ)-structure map then any submodule N and quotient module M/N also have (D,φ)-structure maps. The converse of the proposition also holds.*

Proof. Let D' be a (D,φ)-structure map on M. Then for the submodule N, D' is a (D,φ)-structure map on N, too. For the module M/N define E' by $E'(\mathrm{m}+N) = D'(m) + N$. Then E' is a (D,φ)-structure map,

since
$$\begin{aligned} E'((m+N)a) &= E'(ma+N) = D'(ma) + N \\ &= D'(m)\varphi(a) + mD(a) + N \\ &= (D'(m) + N)\varphi(a) + mD(a) \\ &= E'(m+N)\varphi(a) + mD(a). \end{aligned}$$

The converse is trivial.

This short proposition has many implications as follows. Combining proposition 1 and 2, we can get.

COROLLARY 3. *Let M be a right A-module and let N be a submodule of M.*

Then M is a $A[X, D, \varphi]$-module if and only if both N and M/N are right $A[X, D, \varphi]$-moules.

COROLLARY 4. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Then M is a $A[X, D, \varphi]$-module if and only if both M' and M'' are $A[X, D, \varphi]$-modules.*

COROLLARY 5. *Let M be a right A-module, and let N and N' be submodules of M. If $M=N+N'$ and if N and N' are $A[X, D, \varphi]$-modules, then so is M.*

COROLLARY 6. *A finite direct sum of $A[X, D, \varphi]$-modules is a $A[X, D, \varphi]$-module.*

PROPOSITION 7. *Let M and N be right A-modules and let M be a right $A[X, D, \varphi]$-module also. If $f: M \to N$ is a surjective φ-map, then N is a $A[X, D, \varphi]$-module also.*

Proof. Let D' be a (D, φ)-structure map on M. Now define a (D, φ)-structure map E' on N by $E'(na) = E'(f(m)a) = fD'(ma)$, for all n in N and a in A, where m is an element in M such that $f(m)=n$. This is well-defined by the linearity of D'. Now we check such E' is a (D, φ)-structure map on N. For all n in N and a in A

$$\begin{aligned} E'(na) &= E'(f(m)a) = fD'(ma) \\ &= f(D'(m)\varphi(a) + mD(a)) \\ &= f(D'(m)\varphi(a)) + f(m)D(a) \\ &= E'f(m)\varphi(a) + f(m)D(a) \\ &= E'(n)\varphi(a) + nD(a). \end{aligned}$$

COROLLARY 8. *If $f: M \to N$ is a surjective D-map, if M is a $A[X, D]$-module, then N is also.*

Proof. Take φ = identity map in Proposition 7.

IV. Generalities on ordinary and φ-differential modules

In this section, let A be a commutative ring with a φ-derivation D, and $\mathcal{D}$ is a ring of differential polynomials in φ-derivation D over A by the usual addition and multiplication induced by the relation $Da = \varphi(a)D + D(a)$ for all a in A. Such $\mathcal{D}$ is called a ring of φ-differential polynomials in D over A. Any left $\mathcal{D}$-module is called a φ-differential module. Note that such an M is necessarily an (A, A)-bimodule and $(\mathcal{D}, A)$-bimodule as well. The action of D on an element m in M is written $D(m)$, moreover $D(ma) = mD(a) + D(m)\varphi(a)$ for all a in A. If we take φ to be an identity map, φ-differential module is usually called differential module ([2], [3]).

For two differential modules M, N we can define a tensor product of M and N over A, $M \underset{A}{\otimes} N$, to be a differential module as follows ([2]), $D(m\otimes n)=D(m)\otimes n+m\otimes D(n)$, for all D in $\mathcal{D}$. Moreover $\mathrm{Hom}_A(M, N)$ also differential module, if M and N are differential modules, by the definition in ([2]).

Now we consider the generalities of φ-differential modules, so may assume M, N as φ-differential modules at first.

PROPOSITION 9. *Under the above assumption, $Hom_A(M, N)$ is a φ-differential module.*

Proof) For f in $\mathrm{Hom}_A(M, N)$, define $(fa)(m)=f(m)a$, for all a in A and m in M. And for all D in $\mathcal{D}$, define $D*f(m)=Df-\varphi*fD$, for all f in $\mathrm{Hom}_A(M, N)$ where $\varphi*f$ is a module homomorphism defined by $\varphi*f(ma)=f(m)\varphi(a)$. Then it is easy to check that $D*f$ is an element of $\mathrm{Hom}_A(M, N)$. Moreover $D*(fa)(m)=D(fa)(m)-\varphi*(fa)D(m) =D(f(m)a)-\varphi*(fD(m)a)=D(f(m))\varphi(a)+f(m)D(a)-fD(m)\varphi(a) = (Df(m)-fD(m))\varphi(a)+f(m)D(a)=D*f(m)\varphi(a)+f(m)D(a)$.

V. φ-differential polynomial rings

In this section we consider the φ-differential polynomial ring $A[X, D, \varphi]$ itself for the special automorphism $\varphi: A\to A$ which is not a constant 1-map and $\varphi(a)-a$ is 0 or invertible for all a in A. With the help of the following lemmas, we extend a kind of Hilbert basis theorem on $A[X, D]$ and $A[X, D, \varphi]$

LEMMA 10 [1]. *For every non-trivial ideal L of a ring A, we have $L\oplus\varphi(L)=A$.*

LEMMA 11. *If L is non-trivial ideal of a ring A, then L is a maximal ideal of A.*

Proof. If $L\subseteq L'\subsetneq A$, then by the lemma 10, for all a in L', there are b and c in L such that $a=b+\varphi(c)$, $\varphi(c)=a-b\in L\cap\varphi(L)$ so $c=0$, thus $a=b$. This meas $L=L'$.

Now, let I be any left ideal of φ-differential polynomial ring $A[X, D, \varphi]$. If I contains a polynomial of degree n, define $L_n(I)$ to be the set of 0 and the elements a in A that appear as a coefficient of X^n of a polynomial in I having degree n. Then $L_n(I)$ is a left ideal of A.

LEMMA 12 [1]. *If I and J are ideals of a ring $A[X, D, \varphi]$ such that $I\subseteq J$ and if $L_i(I)=L_i(J)$ for $i=1, 2, 3, \dots$, then $I=J$.*

THEOREM A. *Let A be a Noetherian ring with a φ-derivation which is not a constant 1-map and $\varphi(a)-a$ is 0 or invertible for all a in A, then the φ-differ-*

ential polynomial ring $A[X, D, \varphi]$ is a left Noetherian ring.

Proof. Let I be a non-trivial left ideal of $A[X, D, \varphi]$, and let $L_n(I)$ be as above remark. If $t=\sum_{i=0}^{n} a_iX^i$ in I has degree n, then $Xt=X(\sum_{i=0}^{n} a_iX^i)=\varphi(a_n)X^{n+1}+\sum_{i=0}^{n}\varphi(a_i)X^i+\sum_{i=0}^{n}D(a_i)$. By the assumption, if $\varphi(a)-a=0$, we have

$$L_n(I)\subseteq L_{n+1}(I)\cdots\cdots \tag{1}$$

Otherwise, $\varphi(a)-a$ is invertible, by the lemma 11 each $L_n(I)$ is *a* maximal ideal of A, in which case there is no problem to prove the theorem.

Now let

$$I_0\subseteq I_1\subseteq I_2\subseteq\cdots\subseteq I_n\subseteq\cdots\cdots \tag{2}$$

be an ascending sequence of left ideals of $A[X, D, \varphi]$.
For $i\leq j$, $I_i\subseteq I_j$ implies

$$L_q(I_i)\subseteq L_q(I_j)\cdots\cdots \tag{3}$$

Now we consider an ascending sequence $L_0(I_0)\subseteq L_1(I_1)\subseteq\cdots\subseteq L_n(I_n)\subseteq\cdots$. Since A is left Noetherian, there exists q such that $L_i(I_j)=L_q(I_q)$ for all $i\geq q$ and $j\geq q$. For fixed i, look at the sequence $L_i(I_0)\subseteq L_i(I_1)\subseteq\cdots\subseteq L_i(I_n)\subseteq L_i(I_{n+1})\subseteq\cdots\cdots$, there exists $n(i)$ such that $L_i(I_j)=L_i(I_{n(i)})$ for all j, $n(i)$, the integer $n(i)$ is bounded, snince ring A is Noetherian, say by n_0. And then $L_i(I_j)=L_i(I_{n_0})$ for $i=1,2,3\cdots$, by the lemma 12, we have $I_j=I_{n_0}$ for all $j\geq n_0$. Thus in (2), the ascending sequence is stopped.

COROLLARY B. *If A is a Noetherian ring. Then differential polynomial ring, $A[X, D]$, is Noetherian ring.* [2].

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