PUSAN KYŎNGNAM MATHEMATICAL JOURNAL
Vol. 1, 101~112, 1985

OPTIMAL BOUNDARY CONTROL PROBLEMS FOR SYMMETRIC HYPERBOLIC SYSTEMS WITH CHARACTERISTIC BOUNDARIES.

SUNG KAG CHANG

1. Introduction.

A number of authors [C. 1, C. 2, L. 1, L. 2, L. 3, V. 1] known to us have studied optimal boundary control problems for hyperbolic systems in several variables *with noncharacteristic boundaries.*

But ~~many important problems~~, for instance, Maxwell equations and linearized shallow water equations, have characteristic boundary conditions.

In this paper, we study control problems for hyperbolic systems with characteristic boundaries, which is different from others.

Let Ω be an open domain in R^m with smooth boundary Γ for an integer $m>1$. We consider a first order differential operator of the form

$$(1.1)\qquad A(x, \partial/\partial x)=\sum_{j=1}^{m} A_j(x)\frac{\partial}{\partial x_j}+C(x)\ \text{ for } x\in\Omega$$

where $A_j(x)$ and $C(x)$ are $(l+n)\times(l+n)$ smooth symmetric matrix valued functions on $\bar{\Omega}$, l and n are given positive integers.

We also require that $A_j(x)$ and $C(x)$ are constant for sufficiently large $|x|$ in $\bar{\Omega}$.

We assume the uniform characteristic boundary, that is, (1.2) the normal matrix $N_A(x)=\sum_{j=1}^{m} A_j(x)n_j(x)$ have constant rank n for all $x\in\Gamma$ where $(n_j(x))$ are the inward normals to the boundary Γ at x in Γ.

Without loss of generality, we may assume that $N_A(x)$ have k negative real eigenvalues and $(n-k)$ positive real eigenvalues.

We now consider a mixed initial boundary value problem as

$$(1.3)\quad \begin{cases} \dfrac{\partial y}{\partial t}=A(x,\ \partial/\partial x)y+h & \text{on } [0, T]\times\ \Omega=Q \\ \beta(x)y=u & \text{on } [0, T]\times\ \Gamma=\Sigma \\ y(0)=f & \text{on } \Omega, \end{cases}$$

where $\beta(x)$ is a boundary operator ~~which annihilates~~ the null space of the normal matrix $N_A(x)$ at $x\in\Gamma$, $h\in L^2(Q)$, $f\in L^2(\Omega)$and $u\in L^2(\Sigma)$(*).

For the simplicity, we transform our problem into one on a half-space by using local coordinate changes and a partition of unity. Thus we may assume without loss of generality that

(1.4) $\Omega=\{x\in R^m\ |x_1>0\}$ and $\Gamma=\{x\in R^m\ |x_1=0\}$.

Then the normal matrix $N_A(x)=A_1(x)$ for x in Γ.

By smooth change of coordinates, we may assume

$$(1.5)\quad A_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & A_n^- & 0 \\ 0 & 0 & A_n^+ \end{pmatrix}, \qquad A_n=\begin{pmatrix} A_n^- & 0 \\ 0 & A_n^+ \end{pmatrix}$$

where A_n^- is a negative-definite $k\times k$ matrix and A_n^+ is positive-definite $(n-k)\times(n-k)$ matrix.

(*) $L^2(Q)=L^2(Q;R^{l+n})$, $L^2(\Omega)=L^2(\Omega;R^{l+n})$ and $L^2(\Sigma)=L^2(\Sigma;R^k)$.

Henceforth, we assume (1.4) and (1.5) without loss of generality.

For $x \in R^m$, $y \in R^{l+n}$, we partition x and y as $x=(x_1, x^1)^T$, $y=(y_0, y_-, y_+)^T$ and $y_n=(y_-, y_+)^T$ where $x^1=(x_2, \cdots\cdots x_m)^T$, $y_0=(y_1, \cdots\cdots y_l)^T$, $y_-=(y_{l+1}, \cdots\cdots\cdots y_{l+k})^T$ and $y_+=(y_{l+k+1}, \cdots\cdots\cdots y_{l+n})^T$.

In order to be well-posed for the problem (1.3), it is well known that the boundary condition $\beta(x)y=u$ can be written in the following form;

(1.6) $\qquad y_-=N(x)y_+ + u$ for $\quad x \in \Gamma$,

where $N(x)$ is a smooth $k \times (n-k)$ matrix valued function on Γ which is constant for sufficiently large $|x|$ on Γ (see [M. 1]).

We partition matrices $A_j(x)$ as $\begin{pmatrix} A_j^{11} & A_j^{12} \\ A_j^{21} & A_j^{22} \end{pmatrix}$ where A_j^{11} and A_j^{22} are $l \times l$ and $n \times n$ square matrices respectively for $j=2, \cdots\cdots, m$, and $A_j^{12}=(A_j^{21})^T$ is $l \times n$ matrix.

Let us denote $W_{11}(iw)=\sum_{j=2}^{m} A_j^{11}\, iw_j$, $W_{12}(iw)=\sum_{j=2}^{m} A_j^{12}\, iw_j$ and $W_{22}(iw)=\sum_{j=2}^{m} A_j^{22}\, iw_j$ for $w=(w_2, \cdots w_m) \in R^{m-1}$, $w \neq 0$.

We assume for $W_{11}(iw)$ that

(1.7) the matrix $W_{11}(iw)$ has distinct eigenvalues (pure imaginary) for every $w \in R^{m-1}$, $w \neq 0$.

We assume, without detail, other appropriate conditions for the problem (1.3) to be well-posed in the Kreiss' sense (we refer to [M. 1])

Under appropriate assumptions [M. 1], we have the following theorem.

THEOREM 1. For any $T>0$, $f \in L^2(\Omega)$, $h \in L^2(Q)$ and $u \in L^2(\Sigma)$, the problem (1.3) has a unique strong solution y

in $L^2(Q)$ and unique strong solution $y(t)$ in $L^2(\Omega)$ at each time $t\in[0, T]$.

Moreover y_n has strong boundary value in $L^2(\Sigma)$ and the following inequality holds;

$$(1.8)\qquad |y(t)|_\Omega+|y|_Q+|y_n|_\Sigma\leq C[|f|_\Omega+|h|_Q+|u|_\Sigma],$$

where C is a constant independent on f, h and u.

REMARK. y_0 in theorem 1 may not have boundary value at all in $L^2(\Sigma)$.

We are interested in boundary control problem. Hence here after it may be assumed that f and h are fixed, moreover let $h=0$, and $C(x)=0$.

We now are in a position to formulate optimal control problem for the system (1.3) as follows.

Suppose that F and G be given bounded, self-adjoint and positive-definite linear operators on $L^2(\Omega)$.

A quadratic functional cost J is defined as

$$(1.9)\qquad J(u)=|u|_\Sigma^2+(y, Fy)_Q+(y(T),\ Gy(T))_\Omega$$

for $u\in L^2(\Sigma)$ and correpponding solution y to the system (1.3).

$(C.P)$ Our problem is to minimize the cost $J(u)$ over $u\in L^2(\Sigma)$.

Our main goal is to show that an optimal control u^0 exists in $L^2(\Sigma)$ and it can be synthesized as a feedback form, that is,

$$u^0(t)=BP(t)y^0(t)\quad \text{for } a.e.\ t\in[0, T]$$

where B is an unbounded linear operator on $L^2(\Omega)$ into $L^2(\Gamma)$, $P(t)$ is a Riccati operator on $L^2(\Omega)$ and y^0 is the optimal trajectory corresponding to u^0.

We state the main results in the next section and sketch briefly their proofs in section 3.

2. Main Results

We introduce an operator A on $L^2(\Omega)$ as

(2.1) $$Ay = A(x, \partial/\partial x)y \text{ for } y \in D(A)$$

where the domain $D(A) = \{y \in L^2(\Omega) \mid Ay \in L^2(\Omega) \text{ and } y_- = Ny_+ \text{ on } \Gamma\}$. Then it is easily seen that A is closed and densely defined on $L^2(\Omega)$, furthermore it generates a strongly continuous semigroup $S(t)$ on $L^2(\Omega)$.

REMARK. The adjoint A^* of A is given by

(2.2) $$A^*y = -\sum_{j=1}^{m} A_j(x) \frac{\partial}{\partial x_j} y - \sum_{j=1}^{m} \left(\frac{\partial}{\partial x_j} A_j(x)\right) y$$

for $y \in D(A^*)$ where the domain

$$D(A^*) = \{y \in L^2(\Omega) \mid A^*y \in L^2(\Omega)$$

and $y_+ = -(A_n^+)^{-1} N^T A_n^- y_-$ on $\Gamma\}$.

It is also well-known that A^* generates the adjoint semigroup $S^*(t)$ of $S(t)$ on $L^2(\Omega)$.

In order to introduce a Dirichlet map "D" which extends boundary functions to interior functions in a certain way, we consider the following boundary value problem:

(2.3) $$\begin{cases} A(x, \partial/\partial x)y = Ky \text{ on } \Omega \\ y_- = Ny_+ + u \text{ on } \Gamma \end{cases}$$

for $u \in L^2(\Gamma)$ where K is a large constant.
Then we have the following lemma.

LEMMA 2.1.

The problem (2.3) is well-posed for a sufficiently large number $K>0$. Moreover, the following inequality holds:

$$(2.4) \qquad |y|_{\Omega}+|y_n|_{\Gamma}\leqslant C|u|_{\Gamma} \quad \text{for} \quad u\in L^2(\Gamma)$$

where C is a constant independent on u.

Once we have chosen K so that (2.3) is well-posed, we fix K. For the simplicity, we may assume $K=0$ (If $K\neq 0$, we simply translate A by K)

From lemma 2.1, we define the Dirichlet map D as:

$Du=y$ if y is the solution to (2.3).

Then D is a bounded linear operator on $L^2(\Gamma)$ into $L^2(\Omega)$.

Now have the following trace operator.

LEMMA 2.2.

$$D^*A^*y=A_n^-y.|_{\Gamma} \quad \text{for} \quad y\in D(A^*).$$

Let us define an operator L on $L^2(\Sigma)$ as

$$(2.5) \quad (Lu)(t)=A\int_0^t S(t-s)\ D\ u(s)ds \qquad \text{for } 0\leqq t\leqq T$$

and $u\in D(L)$ which is a subspace of $L^2(\Sigma)$.

We have a semigroup representation of the solutions to the mixed problem(1.3).

THEOREM 2.3. (Semigroup Representation).

(1) the operator L is a bounded linear operator on $L^2(\Sigma)$ into $C([O, T];L^2(\Omega))$,

(2) the solution y to the problem (1.3) is given by

$$(2.6) \qquad y(t)=S(t)f-(Lu)(t) \qquad \text{for } 0\leqslant t\leqslant T.$$

From theorem 2.3, it is easily seen that our control problem $(C.P)$ has a unique optimal control u^0 in $L^2(\Sigma)$ by standard argument (see [C.1, L.1]).

Let us denote $\Phi(t,s)$ the evolution operator which describes the evolution of the optimal trajectory y^0 with initial time $s, 0 \leqq s \leqq t \leqq T$ (see [C. 1, L. 1]). Then we have the following synthesis of optimal control u^0.

THEOREM 2.4. (Feedback synthesis)

The optimal control u^0 can be written in the form $u^0(t) = D^*A^*P(t)y^0(t)$ a.e. $t \in [0, T]$ where $P(t)$ is a bounded self-adjoint and positive-definite operator on $L^2(\Omega)$ which satisfies the following Riccati equation:

(R.E.1) for x, $y \in L^2(\Omega)$ and $0 \leq t \leq T$,

$$(P(t)x, y)_\Omega = \int_t^T (\Phi(s,t)x, F\,\Phi(s,t)y)_\Omega\, ds$$
$$+ \int_t^T (D^*A^*P(s)\,\Phi(s,t)x, D^*A^*P(s)\Phi(s,t)y)_\Gamma\, ds$$
$$+ (\Phi(T,t)x, G\,\Phi(T,t)y)_\Omega.$$

For a moment, we assume that

(2.7) F and G map $L^2(\Omega)$ into $D(A^*)$.

We denote $\mathscr{P}$ the class of one parameter families of operators $P(t)$ on $L^2(\Omega)$ which are self-adjoint, positive-definite and satisfy the following conditions;

(2.8.1) $D^*A^*\bar{P}(\cdot): L^2(\Omega) \to L^\infty([0, T]; L^2(\Gamma))$ are bounded and

(2.8.2) $D^*A^*\bar{P}(\cdot)S(\cdot)AD: L^2(\Gamma) \to L^2(\Sigma)$ are bounded.

REMARK. Under assumption (2.7), it is shown that the operator $P(t)$ in theorem 2.4 is in the class $\mathscr{P}$ (see [C. 1]).

Then we have the following theorem.

THEOREM 2.5. Under assumption (2.7), the operator $P(t)$ in theorem 2.4 is the unique solution in the class $\mathscr{P}$ to the following Riccati equation

$(R.E.2)$ for $x, y \in L^2(\Omega)$ and $0 \leq t \leq T$,

$$(P(t)x,\ y)_\Omega = \int_t^T (S(s-t)\ x,\ FS(s-t)y)_\Omega\ ds$$
$$- \int_t^T (D^*A^*P(s)S\ (s-t)\ x,\ D^*A^*P(s)S$$
$$(s-t)y)_\Gamma\ ds + (S(T-t)x,\ GS(T-t)y)_\Omega,$$

(R.E.3) for $x, y \in D(A)$, a.e. t in $[0, T]$,

$$\frac{d}{dt}(P(t)x, y)_\Omega = -(x, Fy)_\Omega - (P(t)Ax, y)_\Omega$$
$$-(P(t)\ x, Ay)_\Omega$$
$$+(D^*A^*P(t)x,\ D^*A^*P(t)y)_\Gamma$$

with terminal condition $P(T) = G$.

REMARK. without smoothness assumption(2.7), we are not sure whether $D^*A^*P(t)x$ are well-defined in L^2 -sense for $x \in L^2(\Omega)$.

Suppose that the pairs F_n and G_n satisfy assumption (2.7) for all n=1, 2, …. Let $P_n(t)$ be the corresponding Riccati operators to the pairs F_n and G_n, for n=1, 2, 3, ….

Now we do not assume any smoothness for F and G. Then we have the following convergence.

THEOREM 2.6. Suppose that $F_n \to F$ and $G_n \to G$ strongly on $L^2(\Omega)$ as $n \to \infty$. Then $P_n(t) \to P(t)$ uniformly in $t \in [0, T]$, strongly on $L^2(\Omega)$ as $n \to \infty$.

Moreover, for $x, y \in L^2(\Omega)$, $0 \leq t \leq T$,

$$(P(t)x,\ y)_\Omega = \int_t^T (S(s-t)x,\ FS(s-t)y)_\Omega\ ds + (S(T-t)x,$$
$$GS(T-t)y)_\Omega - \lim_{n\to\infty} \int_t^T (D^*A^*P_n\ (s)\ S\ (s-t)x,$$
$$D^*A^*P_n(s)S(s-t)y)_\Gamma\ ds.$$

3. Proofs of Results

We sketch the proofs of lemmas 2.1 and 2.2 briefly, and we omit proofs of theorems 2.3, 2.4, 2.5 and 2.6 since their proofs are similar to those in [c.1].

Suppose for a moment that the coefficients $A_i(x)$ and $N(x)$ in (2.3) are frozen at the values on a boundary point $x_0^1 \in \Gamma$.

Then we apply Fourier transform the equation (2.3) in the tangential variables x^1, and denote the transforms of y, y_0 and y_n by $\hat{y}$, $\hat{y}_0$ and $\hat{y}_n$ respectively.
We arrive at

$$(3.1)\quad \begin{cases} A_1\ (0, x_0^1)\dfrac{d\hat{y}}{dx_1} = (K - W(0,\ x^0{}_1,\ iw))\hat{y} \text{ for } x_1 > 0 \\ \hat{y}_- = N(x_0^1)\hat{y}_+ + \hat{u} \qquad\qquad \text{for } x_1 = 0 \end{cases}$$

where $W(0,\ x^1, iw) = \begin{pmatrix} W_{11}(0, x_0^1,\ iw) & W_{12}(0, x_0^1,\ iw) \\ W^T{}_{12}(0, x_0^1,\ iw) & W_{22}(0, x_0^1,\ iw) \end{pmatrix}$

and $\quad w = (w_2, \cdots\cdots,\ w_m) \in R^{m-1},\ w \neq 0.$
By assumption (1.5) and (1.7), we have

$$(3.2)\quad \begin{cases} \hat{y}_0 = [K - W_{11}\ (0, x_0^1, iw)]^{-1} W_{12}(0, x_0^1, iw)\hat{y}_n, \quad x_1 > 0 \\ A_n \dfrac{d\hat{y}_n}{dx_1} = [W^T{}_{12}\ (K - W_{11})^{-1}\ W_{12} + (K - W_{22})]\hat{y}_n, \\ \qquad\qquad x_1 > 0 \\ \hat{y}_- = N\hat{y}_+ + \hat{u} \qquad \text{for } x_1 = 0. \end{cases}$$

We may rewrite (3.2) in a pseudo-differential form for the variable coefficient problem as

$$(3.3.1)\quad \hat{y}_0 = [K - \overline{W}_{11}(x, iw)]^{-1}\overline{W}_{12}\ (x, iw)\hat{y}_n\ , x_1 > 0$$

$$(3.3.2)\quad A_n(x)\,\frac{d\hat{y}_n}{dx_1} = [\overline{W}^T{}_{12}\ (x, iw)(K - \overline{W}_{11}(x, iw))^{-1}\overline{W}_{12}\ (x, iw) + (K - \overline{W}_{22}(x, iw))]\hat{y}_n,\ x_1 > 0$$

(3.3.3) $\hat{y}_- = N(x^1)\hat{y}_+ + \hat{u}, x_1 = 0$ where
$W_{11}(x, iw)$, $W_{12}(x, iw)$ and $W_{22}(x, iw)$ are the pseudo-differential operators of order 1 corresponding to the differential operators $\sum_{j=2}^{m} A_j^{11}(x)\frac{\partial}{\partial x_j}$, $\sum_{j=2}^{m} A_j^{12}(x)\frac{\partial}{\partial x_j}$ and $\sum_{j=2}^{m} A_j^{22}(x)\frac{\partial}{\partial x_j}$ respectively.
Let $\bar{M}(K, x, iw) = A_n^{-1}[W^T{}_{12}(K - W_{11})^{-1} W_{12} + K - W_{22}]$.

Then we arrive at

(3.4.1) $\hat{y}_0 = (K - W_{11})^{-1} W_{12}\hat{y}_n \qquad x_1 > 0$

(3.4.2) $\dfrac{d\hat{y}_n}{dx_1} = \bar{M}\hat{y}_n \qquad x_1 > 0$

(3.4.3) $\hat{y}_- = N\hat{y}_+ + \hat{u} \qquad x_1 = 0.$

The problems
(3.4.2) and (3.4.3) are the same kind Majda and Osher studied in [M. 1].

That is, they showed that there exists a symmetrizer of $\bar{M}$ whose symbol $\bar{R}$ (K, x, iw) is of order zero and satis fies the following properties (see [M. 1]);

(3.5.1) $\bar{R}$ is Hermitian,

(3.5.2) $v^T\bar{R}v \geqq \delta|v|^2 - \varepsilon|g|^2$ for all vectors satisfying the boundary condition $v_- = N v_+ + g$,

(3.5.3) Re $(\bar{R}\bar{M}) \geqq \delta$ where $\delta > 0$ and $\varepsilon > 0$ are constants independent on $x \in \Omega$, $w \in R^{m-1}$ and $K > 0$ large enough.
Thus combining (3.4.2) and (3.4.3) with (3.5.1), (3.5.2) and (3.5.3), we have, for sufficiently large $K > 0$,

(3.6) $|y_n|_\Omega + |y_n|_\Gamma \leqq C|u|_\Gamma$ where C is a constant independent on u.
On the other hand, we suppose that y is a solution to (2.3).

We take the inner product (2.3) with y on Ω.
Then we have

(3.7) $$(A(x, \partial/\partial x)y,\ y)_\Omega = K|y|^2_\Omega.$$

By Green's formula, the left hand side of (3.7) becomes

$$-\frac{1}{2}(A_n\, y_n,\ y_n)_\Gamma - \frac{1}{2}(y, \sum_{j=1}^{m} (\frac{\partial}{\partial x_j} A_j)y)_\Omega.$$

That is,

$$K|y|^2{}_\Omega + \frac{1}{2}(y, \sum_{j=1}^{m} (\frac{\partial}{\partial x_j} A_j)y)_\Omega = -\frac{1}{2}\ (A_n\, y_n,\ y_n)_\Gamma.$$

Thus, for sufficiently large $K>0$, we arrive at

(3.8) $|y|_\Omega \leq C\ |y_n|_\Gamma$ for a constant C.

From (3.6) and (3.8), we derive the ineguality $|y|_\Omega + |y_n|_\Gamma \leqq C|u|_\Gamma$ for sufficiently large $K>0$.
Once we have the energy inequality, we can deduce the uniqueness and existence easily as in [C. 1].
This completes the proof of lemma 2.1.

We assume, without loss of generality, $K=0$.
From Green's formula, we have for $y \in D(A^*)$ and $g \in L^2(\Gamma)$,

(3.9) $$(A^*y, Dg)_\Omega = (y, A(x, \frac{\partial}{\partial x})\ Dg)_\Omega + (A_n y_n, (Dg)_n)_\Gamma.$$

By the definition of the operator D, (3.9) becomes

$$(A^*y, Dg) = (A_n y_n,\ (Dg)_n\ )_\Gamma.$$

From the fact that $y \in D(A^*)$ and $(Dg)_- = N(Dg)_+ + g$ on Γ, we arrive at

$$(A^*y, Dg)_\Omega = (A_n y_-, g)_\Gamma$$

which implies lemma 2.2..

References

[B.1] A.V. Balakrishnan, Applied functional analysis, Springer-Verlag, 2nd ed., NewYork, 1981.

[C.1] S. Chang, Riccati equations for nonsymmetric and nondissipative hyperbolic systems with L^2-boundary controls, Ph. D. thesis, Math. Depart., University of Florida, 1984.

[C.2] R. Curtain and A. Pritchard, An abstract theory for unbounded control action for distributed parameter systems, SIAM J. Control and Opt. 15, 1977, pp. 566–611.

[K.1] T. Kato, Perturbation theory of linear operators, Springer-Verlag, 2nd ed., New York, 1980.

[K.2] H.O. Kreiss, Initial boundary value problems for hyperbolic systems, Comm. Pure. and Appl. Math., 13, 1970, pp. 277–298.

[L.1] I. Lasiecka and R. Triggiani, Riccati equations for hyperbolic partial differential equations with L^2-Dirichlet boundary terms, manuscript, 1983.

[L.2] J.L. Lions, Optimal control of systems governed by partial diff. equations, Springer-Verlag, 1971.

[L.3] J.L. Lions, Nonhomogeneous boundary Value problems and applications, Vol. II, Springer-Verlag, 1972.

[M.1] A. Majda and S. Osher, Initial boundary Value problems for hyperbolic equations with uniformly characteristic boundary, Comm. pure & appl. Math. 17, 1975, pp. 607–675.

[V.1] R. Vinter and T. Johnson, Optimal control of nonsymmetric hyperbolic systems in n-variables on the half-space, SIAM J. Control and Opt. 15, 1977, pp. 129–143.