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A NOTE ON WEAKLY IRRESOLUTE MAPPINGS

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I. Introduction.

In 1963, N. Levine introduced a class of semi-continuous mappings which properly contains the class of all continuous mappings and in [5], the notion of an irresolute mapping which is stronger than that of semi-continuity, but is independent of that of continuity was introduced. The concept of weakly irresolute mappings was introduced in [2, Definition 6]. In this note, it will be shown that the class of weakly irresolute mappings properly contains that of irresolute mapping [5], and it is independent of that of semi-continuous mappings [6], of that of almost irresolute mappings [8] and of that of set-s-connected mappings [3]. Further, characterizations and some basic properties of weakly irresolute mappings are investigated.

Throughout this paper, spaces mean topological spaces on which no separation axioms are assumed and $f:X \to Y$ denotes a mapping from a space X into a space Y. Let A be a subset of a space X. By $T(X)$, $cl_X(A)$ and $\mathrm{int}_X(A)$ (T, $cl(A)$ and $\mathrm{int}(A)$ without confusions) we will denote, respectively, the topology on X, the closure of A and the interior of $A \subset X$. A set A is semiopen [6] in a space X if there exists an $0 \varepsilon T(X)$ such that $0 \subset A \subset cl(0)$, and is semiclosed [8] iff its complement is semiopen. The intersection of all the semiclosed sets containing A is called the semiclosure [9] of A and the union of all the semiopen sets contained in $A\subset X$ is called the semi-interior [9] of A. By $SO(X)$, scl(A) and sint(A) we will denote, respectively, the family of all semiopen sets in a space X, the semi-closure of A and the semi-interior of $A\subset X$. It was shown in well-known papers that int$(A)\subset$sint$(A)\subset A\subset$scl$(A)\subset$cl(A); $A\subset B$ implies sint$(A)\subset$ sint(B) and scl$(A)\subset$scl(B); A is semiopen (resp. semiclosed) iff $A=$ sint(A) (resp. $A=$scl(A)). A set N of a space X is called a semi-neighborhood (written semi-nbd) [1] of a point $x \varepsilon X$ if there exists an $U \varepsilon SO(X)$ such that $x \varepsilon U\subset N$. It was shown that $A \varepsilon SO(X)$ iff A is the semi-nbd of each of its points [1]. A point $p \varepsilon X$ is termed a semi-limit point of A [7] iff, for each $U \varepsilon SO(X)$ containing p, $U\cap(A-\{p\})\neq\phi$. The union of A and sd(A), where sd(A) denotes the set of all the semi-limit points of A, called the semi-drived set of A, is equal to scl(A). A is semiclosed iff A contains sd(A).

A mapping f: $X\to Y$ is said to be irresolute [5] iff for every $V \varepsilon SO(Y)$, $f^{-1}(V) \varepsilon SO(X)$ iff for each $x \varepsilon X$ and each semi-nbd V of $f(x)$, there exists a semi-nbd U of x in X such that $f(U)\subset V$. A mapping f: $X\to Y$ is said to be semi-continuous [6] iff for every $V \varepsilon T(Y)$, $f^{-1}(V) \varepsilon SO(X)$. Every irresolute mapping is semi-continuous but not conversely [5]. A space X is semi-T_2[10] iff for each pair $x, y \varepsilon X$, $x\neq y$, there exist disjoint A, $B \varepsilon SO(X)$ such that $x \varepsilon A$ and $y \varepsilon B$ iff for each pair $x, y \varepsilon X$, $x\neq y$, there exists an $U \varepsilon SO(X)$ such that $y \varepsilon U$ and $x \notin$ scl(U). By a semi-clopen set we mean a set which is both semiopen and semiclosed. A space X is s-connected [11] iff no nonempty proper subset of X is semi-clopen; hence every indiscrete space is s-connected. A subset of a space X is s-connected iff it is s-connected as a subspace of X.

II. Weakly irresolute mappings.

DEFINITION 1. A mapping $f:X\rightarrow Y$ is said to be weakly irresolute [2] if for each $x \varepsilon X$ and each semi-nbd $V\subset Y$ of $f(x)$, there exists a semi-nbd U of x such that $f(U)\subset \mathrm{scl}(V)$.

We now give a characterization of weakly irresolute mappings.

THEOREM 1. A mapping $f: X\rightarrow Y$ is weakly irresolute iff for each $0 \varepsilon SO(Y)$, $\subset \mathrm{sint}(f^{-1}(\mathrm{scl}(0)))$.

PROOF. Let $x \varepsilon f^{-1}(0)$. Then $f(x) \varepsilon 0$. Thus, by Definition 1, there exists a $G \varepsilon SO(X)$ containing x such that $f(G)\subset \mathrm{scl}(0)$. This implies $x \varepsilon G \subset f^{-1}(\mathrm{scl}(0))$, i.e., $x \varepsilon \mathrm{sint}(f^{-1}(\mathrm{scl}(0)))$. Conversely, let $x \varepsilon X$ and $f(x) \varepsilon 0 \varepsilon SO(Y)$. Then $x \varepsilon f^{-1}(0)\subset \mathrm{sint}(f^{-1}(\mathrm{scl}(0)))$. Let $G=\mathrm{sint}(f^{-1}(\mathrm{scl}(0)))$. Then $f(G)\subset \mathrm{scl}(0)$. The proof is complete.

It is quite evident that every irresolute mapping is weakly irresolute. A weakly irresolute mapping may fail to be irresolute, as shown by the following example.

EXMPLE 1. Let $X=\{a,b,c,d\}$ with $T(X)=\{\phi, X, \{d\}, \{a,c\}, \{a,c,d\}\}$ and $Y=\{p,q,r\}$ with $T(Y)=\{\phi, Y, \{p\}\}$. Then the mapping $f:X\rightarrow Y$, defined by $f(a)=p$, $f(b)=f(c)=q$ and $f(d)=r$, is, obviously, weakly irresolute but not irresolute. Note that f is also not semi-continuous.

DEFINITION 2. A space X is strongly s-regular iff, for every point $x \in X$ and every semiclosed set F of X such that $x \notin F$, there exist disjoint U, $V \in SO(X)$ such that $x \in U$ and $F \subset V$.

It can be easily shown that a space X is strongly s-regular iff for each point $x \in X$ and each $V \in SO(X)$ containing x, there exists a $U \in SO(X)$ containing x such that $\mathrm{scl}(U) \subset V$.

THEOREM 2. Let $f: X \to Y$ be a weakly irresolute mapping. If Y is strongly s-regular. Then f is irresolute and hence semi-continuous.

PROOF. Let $x \in X$ and $V \in SO(Y)$ with $f(x) \in V$. Since Y is strongly s-regular, there exists an $M \in SO(Y)$ containing $f(x)$ such that $\mathrm{scl}(M) \subset V$. Since f is weakly irresolute, there exists a $U \in SO(X)$ containing x such that $f(U) \subset \mathrm{scl}(M) \subset V$. Thus f is irresolute.

A semi-continuous mapping may fail to be weakly irresolute, as shown by the following example. Therefore, weakly irresolute mappings are, in general, independent of semi-continuities from Example 1.

EXMPLE 2. Let $X=\{a,b,c\}$ with $T(X)=\{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$ and $Y=\{p,q,\mathrm{r}\}$ with $T(Y)=\{\phi, Y, \{p\}, \{q\}, \{p,q\}\}$. Then the mapping $f: X \to Y$, defined by $f(a)=p$, $f(b)=q$ and $f(c)=r$, is semi-continuous but not weakly irresolute.

A mapping f: $X \to Y$ is said to be almost-open [13] if, for every $V \in T(Y)$, $f^{-1}(\mathrm{cl}(V))$ $(\mathrm{cl} \subset (f^{-1}(V))$. It is known that every open mapping is almost-open and a continuous and almost-open mapping is always not open.

LEMMA 1. If $f: X \to Y$ is semi-continuous and almost-open, then f is irresolute.

PROOF. It is easy to prove and is thus omitted.

From Lemma 1, we obtain that a semi-continuous mapping is weakly irresolute if it is almost-open and hence open.

THEOREM 3. If $f: X \to Y$ is weakly irresolute, then scl $(f^{-1}(V)) \subset f^{-1}(\mathrm{scl}(V))$ for each $V \varepsilon SO(Y)$.

PROOF. It is sufficient to show that if $x \varepsilon \mathrm{scl}(f^{-1}(V)) - f^{-1}(V)$, then $x \varepsilon f^{-1}(\mathrm{scl}(V))$. Suppose $x \notin f^{-1}(\mathrm{scl}(V))$, that is, $f(x) \notin \mathrm{scl}(V)$. Then there exists a $W \varepsilon SO(Y)$ such that $f(x) \varepsilon W$ and $W \cap V = \phi$. Since $V \varepsilon SO(Y)$, we have $\mathrm{scl}(W) \cap V = \phi$. Since f is weakly irresolute, there exists an $U \varepsilon SO(X)$ containing x such that $f(U) \subset \mathrm{scl}(W)$. Accordingly, $f(U) \cap V = \phi$. On the other hand, if $x \varepsilon$ scl $(f^{-1}(V))$ and $x \notin f^{-1}(V)$, then $x \in \mathrm{scl}(f^{-1}(V)$, and so we have $U \cap f^{-1}(V) \neq \phi$ so that $f(U) \cap V \neq \phi$. This means a contradiction. Therefore, $x \varepsilon f^{-1}(\mathrm{scl}(V))$. This proves the theorem.

From Theorem 3, it is obvious that if $f: X \to Y$ is weakly irresolute, then $f(\mathrm{scl}(f^{-1}(V))) \subset \mathrm{scl}(V)$ for each V ε $SO(Y)$.

DEFINITION 3. A mapping $f: X \to Y$ is termed almost irresolute [8] if for each point $x \varepsilon X$ and each semi-nbd $V \subset Y$ of $f(x)$, $\mathrm{scl}(f^{-1}(V))$ is a semi-nbd of x.

An almost irresolute mapping need not be weakly irresolute, as show by the following example.

EXMPLE 3. Let $X = Y = \{a, b, c, d\}$ with topologies, $T(X) = \{\phi, X, \{a, d\}, \{c\}, \{a, c, d\}\}$ and $T(Y) = \{\phi, Y, \{a\}, \{b,$

$c\}$, $\{a,b,c\}\}$. Then the mapping f: $X \rightarrow Y$, defined by $f(a)=f(b)=f(c)=a$ and $f(d)=b$, is almost irresolute but not weakly irresolute.

THEOREM 4. If $f:X \rightarrow Y$ is almost irresolute and $\mathrm{scl}(f^{-1}(V)) \subset f^{-1}(\mathrm{scl}(V))$ for each $V \varepsilon SO(Y)$, then f is weakly irresolute.

PROOF. For any point $x \varepsilon X$ and $V \varepsilon SO(Y)$ containing $f(x)$, we have $\mathrm{scl}(f^{-1}(V)) \subset f^{-1}(\mathrm{scl}(V))$ by hypothesis. Since f is almost irresolute, there exists a $U \varepsilon SO(X)$ such that $x \varepsilon U \subset \mathrm{scl}(f^{-1}(V)) \subset f^{-1}(\mathrm{scl}(V))$. Thus $f(U) \subset \mathrm{scl}(V)$.

The converse to Theorem 4 does not hold, in general, as shown by the following example.

EXMPLE 4. Let $X=Y=\{a,b,c,d\}$ with topologies, $T(X)=\{\phi, X, \{a\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ and $T(Y)=\{\phi, Y, \{a\}, \{a,c\}\}$. Then the identity mapping i is weakly irresolute but not almost irresolute.

EXAMPLE. 4. An almost irresolute mapping $f: X \rightarrow Y$ is weakly irresolute iff $\mathrm{scl}(f^{-1}(V)) \subset f^{-1}(\mathrm{scl}(V))$ for each $V \varepsilon SO(Y)$.

PROOF. From Theorem 3 and 4.

DEFINITION 4. Let A be a subset of a space X. The weakly irresolute mapping from X onto a subspace A of X is called a weakly irresolute retraction if the restriction $f|A$ is the identity mapping on A. We call such an A a weakly irresolute retract of X.

LEMMA 2. If A is semiopen and U is open in a space X, then $A \cap U$ is semiopen in X. (Refer to [9]).

THEOREM 5. Let A be a subset of a space X and f:

$X \to A$ be a weakly irresolute retraction of X onto A. If X is T_2 then A is semiclosed in X.

PROOF. Suppose A is not semiclosed. Then there exists a semi-limit point x of A in X such that $x \varepsilon \mathrm{scl}(A)$ but $x \not\varepsilon A$. Since f is weakly irresolute retraction, $f(x) \neq x$. Since X is T_2 there exist disjoint $U, V \varepsilon T(X)$ such that $x \varepsilon U$ and $f(x) \varepsilon V$. Thus $U \cap cl_X(V) = \phi$. Also, $V \cap A \varepsilon T(A)$ and hence $V \cap A \varepsilon SO(A)$ containing $f(x)$. Let $W \varepsilon SO(X)$ with $x \varepsilon W$. Then $U \cap W \varepsilon SO(X)$ contains x, by Lemma 2, and hence $(U \cap W) \cap A \neq \phi$ because $x \varepsilon sd(A)$. Therefore, there exists a point $y \varepsilon (U \cap W \cap A)$. Since $y \varepsilon A$, $f(y) = y \varepsilon U$ and hence $f(y) \not\varepsilon cl_X(V)$. This shows that $f(W) \not\subset cl_X(V)$. Now $cl_A(V \cap A) = cl_X(V \cap A) \cap A \subset cl_X(V)$. Therefore, $f(W) \not\subset cl_A(V \cap A)$ which implies $f(W) \not\subset \mathrm{scl}_A(V \cap A)$. This contradicts the hypothesis that f is weakly irresolute. Thus A is semiclosed in X.

In Theorem 5, X is necessary Hausdorff, as shown by the following example

EXMPLE 5. Let $X = \{a, b, c\}$ with an indiscrete topology and let $A = \{a, b\} \subset X$. Then the mapping $f: X \to A$, defined by $f(a) = a$, $f(b) = f(c) = b$, is weakly irresolute and $f|A$ is the identity mapping on A, that is, A is weakly irresolute retract of X. However, A is not semiclosed in X.

LEMMA 3. A mapping $f: X \to Y$ has a semiclosed graph $G(f)$ [8] if for each $x \varepsilon X$, $y \varepsilon Y$ such that $f(x) \neq y$, there exist $U \varepsilon SO(X)$ and $V \varepsilon SO(Y)$ containing x and y, resepctively, such that $f(U) \cap V = \phi$.

In view of the following example, a weakly irresolute mapping may fail to have a semiclosed graph.

EXMPLE 6. Let $X=\{a,b,c\}$ with an indiscrete topology. Then clearly, the identity mapping i: $X\to X$ is weakly irresolute, but $G(i)$ is not semiclosed.

However, we have the following.

THEOREM 6. If $i:X\to Y$ *is* weakly irresolute and Y is semi-T_2 then $G(f)$ is semiclosed in the product space $X \times Y$.

PROOF. Let $x \in X$ and $y \in Y$ such that $y\neq f(x)$. Then there exists a $V \in SO(Y)$ containing $f(x)$ such that $y \notin$ scl(V), i.e., $y \in (Y-\text{scl}(V)) \in SO(Y)$. Since f is weakly irresolute, there exists an $U \in SO(X)$ containing x such that $f(U)\subset \text{scl}(V)$. Consequently, $f(U)\cap(Y-\text{scl}(V))=\phi$ and so, $G(f)$ is semiclosed, by Lemma 3.

The converse to Theorem 6 need not be true as shown by the following example.

EXMPLE 7. Let $X=\{a,b,c\}$ be the space with $T(X)=\{\phi,X,\{a\},\{b\},\{a,b\}\}$ and $Y=\{a,b,c\}$ be the discrete space. Then the identity mapping i: $X\to Y$ has a semiclosed graph but not weakly irresolute.

LEMMA 4 [11]. A space X is not s-connected iff it is the union of two nonempty disjoint semiopen(respectively, semiclosed) sets.

THEOREM 7. The s-connectedness is invariant under weakly irresolute surjections.

PROOF. Let f: $X\to Y$ be a weakly irresolute surjection on an s-connected space X. Suppose Y is not s-connected. Then exist nonempty disjoint $V_1, V_2 \varepsilon SO(Y)$ such that $V_1 \cup V_2 = Y$. Hence $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$ and their union is X. Since f is surjective, $f^{-1}(V_i) \neq \phi$ for $i=1,2$. By Theorem 1, $f^{-1}(V_i) \subset \text{sint}(f^{-1}(\text{scl}(V_i))$. Since V_i is semiclosed, $f^{-1}(V_i) \subset \text{sint}(f^{-1}(V_i))$. Hence, $f^{-1}(V_i) \varepsilon SO(X)$ for $i=1,2$. This means that X is not s-connected. Contradict.

Exmple 8. Let $X=Y=\{a,b,c\}$. Let X be the indiscrete and Y be the space with $T(Y)=\{\phi, Y, \{a,c\}, \{b,c\}, \{c\}\}$. Let $f: X \to Y$ be given by $f(a)=$ a and $f(b)=f(c)=b$. Then f is weakly irresolute, X is s-connected, but $f(X)=\{a,b\}$, is not s-connected. This example shows that the image of an s-connected set under a weakly irresolute mapping is not necessarily s-connected.

III. Weakly irresolute mappings and set-s-connected mappings.

Lemma 5. [3]. A mapping $f: X \to Y$ is set-s-connected iff for each semi-clopen subset B of $f(X)$, $f^{-1}(B)$ is semi-clopen in X.

The following examples 9 and 10 show that the notion of weakly irresolute mappings is independent of that of set-s-connected mappings.

Exmple 9. Let $X=\{a,b,c\}$ with $T(X)=\{\phi, X, \{a\}, \{a,b\}, \{a,c\}\}$. Then the mapping $f: X \to X$, defined by $f(a)=f(c)=b$ and $f(b)=c$, is weakly irresolute but not set-s-connected.

Exmple 10. Let $X=\{a,b,c,d\}$ with $T(X)=\{\phi, X, \{a,c\}, \{d\}, \{c\}, \{c,d\}, \{a,c,d\}\}$ and $Y=\{a,b,c\}$ with $T(Y)=$

$\{\phi, Y, \{c\}, \{b\}, \{b,c\}\}$. Then the mapping $f:X\to Y$, defined by $f(a)=f(d)=a$ and $f(b)=f(c)=c$, is set-s-connected but not weakly irresolute.

THEOREM 8. If $f:X\to Y$ is weakly irresolute surjection, then f is set-s-connected.

PROOF. Let V be any semi-clopen subset of Y. Since V is semiclosed, $\mathrm{scl}(V)=V$. Thus, by Theorem 1, $f^{-1}(V)\subset \mathrm{sint}(f^{-1}(V))$. Hence $f^{-1}(V)\in SO(X)$. Moreover, by Theorem 3, $\mathrm{scl}(f^{-1}(V))\subset f^{-1}(V)$. Hence $f^{-1}(V)$ is semiclosed in X. Since f is surjective, Lemma 5 f is set-s-connected. It is well-known that, for every space X and each $V\in SO(X)$, $\mathrm{scl}(V)\in SO(X)$ and also $\mathrm{cl}(V)\in SO(X)$.

THEOREM 9. Let X and Y be spaces. If $f:X\to Y$ is set-s-connected surjection, then f is weakly irresolute.

PROOF. Let $x\in X$ and $V\in SO(Y)$ containing $f(x)$. Then $\mathrm{scl}(V)$ is semi-clopen in Y. Since f is set-s-connected surjection, it follows from Lemma 5 that $f^{-1}(\mathrm{scl}(V))=U$ is semi-clopen in X. Therefore, $U\in SO(X)$ containing x such that $f(U)\subset \mathrm{scl}(V)$. Hence f is weakly irresolute.

COROLLARY 2. A surjection $f: X\to Y$ is set-s-connected iff f is weakly irresolute.

PROOF. From Theorem 8 and 9.

COROLLARY 3. If $f: X\to Y$ is a set-s-cennected surjection and Y is semi-T_2 then $G(f)$ is semiclosed in the product space $X\times Y$.

PROOF. From Theorem 6 and 9. In view of Example 7, the converse to Corollary 3 is not true. For, $G(\iota)$ is semiclosed, but ι is not set-s-connected.

Abstract

A mapping f: $X \rightarrow Y$ is introduced to be weakly irresolute if, for each $x \in X$ and each semi-neighborhood V of $f(x)$, there exists a semi-neighborhood U of x in X such that $f(U) \subset \mathrm{scl}(V)$. It will be shown that a mapping f: $X \rightarrow Y$ is weakly irresolute iff(if and only if) $f^{-1}(V) \subset \mathrm{sint}(f^{-1}(\mathrm{scl}(V)))$ for each semiopen subset V of Y. The relationship between mappings described in [3, 5, 6, 8] and a weakly irresolute mapping will be investigated and it will be shown that every irresolute retract of a T_2-space is semiclosed.

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