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# REPRESENTING MEASURES RELATED TO ALGEBRAS

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The theory of representing measures is historically the first subject discussed in the representing theory of complex homomorphism. A major motivation for the study of it stems from an attempt to specialize the Riesz representing theorem to non—selfadjoint algebras. For the function algebra, one of the useful features of boundary problems is that each complex homomorphism of the algebra can be represented as integration with respect to a positive measure. Though there are many methods in solving boundary problems, one of the particular methods is representing measures in complex $C^n$ space.

Recently, K. Hoffman and I. Singer discussed measures and the Silov boundary in paper[1], in particular, H.S. Bear[2,3] foreshowed methods which treat the relationship between any measures and pervasive subalgebras of the maximal function algebra and showed the structure of such measures[4]. W. Rudin[5] proved the representing measure for the ball algebra and connected with Lummer's Hardy space with respect to the annihilating measure in [6].

Now let $B$ be the open unit ball in complex $C^n$ space and $A(B)$ be the ball algebra of $B$ which is the class of complex continuous functions on $S$ (the boundary of $B$) and holomorphic in $B$, then the Hahn-Banach theorem, Riesz representing theorem and properties of the Silov boundary yield the following facts; For any bounded linear functional $\phi$ on $A(B)$, there exists a complex measure $m_\phi$ with respect to $\phi$ such that

$$\phi(f)=\int_S f\,dm_\varphi$$

for all $f$ in $A(B)$ and $\|\phi\|=\|m_\phi\|$, such $m_\phi$ is called a representing measure for $\phi$. If, moreover, it is a probability measure associated linear functional $\phi$ on $A(B)$, $m_\varphi$ has the property $\phi(1)=\|\phi\|=1$. Besides the Silov boundary is the smallest compact Hausdorff space on which the algebra $A$ can be realized as a closed separating algebra of continuous functions.

## 1. The uniqueness of representing measures

Let $X$ be a compact Hausdorff space in $C^n$ and $A$ be the function algebra on $X$. In the case of greatest interest to us, $X$ will be $S$ and $A$ will be $A(S)$, the restriction of the ball algebra $A(B)$ to the boundary $S$ of $B$. Then we have the following facts by the consequence of the maximum modulus theorem.

PROPOSITION 1.1. Two algebras $A(S)$, $A(B)$ are isometrically isomorphic Banach algebra.

PROPOSITION 1.2. Each function algebra $A$ on $X$ is closed in the sup-norm topology, contains the constants and separates points on $X$.

Since $\phi(f)$ is non-negative for any continuous function $f$ on $B$, $0\leq f\leq 1$, and $\|1-f\|\leq 1$, $|\phi(1)-\phi(f)|\leq 1$, so we claim the following fact.

LEMMA 1.3. If $\phi$ is a linear functional on $A(B)$ and

$\phi(1)=\|\phi\|=1$, the associated measure is a probability measure.

It follows that any representing measure for $\phi$ is a probability measure. In above argument we could replace $B$ by any closed subset $B'$ of $B$ such that $|\phi(f)|\leq\sup_{B'}|f|$ for $f$ in subalgebra of $A(B)$. Such a set we call a support set for $\phi$, since it is a set which support a representing measure for $\phi$. The difference of any representing measures for the linear functional $\phi$ on $A(B)$ is always orthogonal to the ball algebra $A(B)$, so the following fact is satisfied.

LEMMA 1.4. Let $m_\phi$, $u_\phi$ be any representing measures for $\phi$, then the difference of $m_\phi$, $u_\phi$ is a real measure on $B$.

Though the uniqueness of representing measures is not guaranteed but we claim the following theorem by the consequence of the above facts and properties of the Silov boundary.

THEOREM 1.5. If there is no non-zero real measure on the Silov boundary which is orthogonal to the ball algebra $A(B)$, then each $\phi$ in the maximal ideal of $A(B)$ has a unique representing measure.

Any algebra is a Dirichlet algebra on $B$ if and only if no non-zero real measure on $B$ is orthogonal to $A(B)$. This implies the following fact.

THEOREM 1.6. If the ball algebra $A(B)$ is a Dirichlet algebra on its Silov boundary, then each linear functional $\phi$ in the maximal ideal of $A(B)$ has a unique representing measure.

PROPOSITION 1.7. Representing measures are unique for every complex homomorphism of $H^p$ for $p=\infty$. Since $H^\infty$ possesses a property which is very close to the Dirichlet property.

## 2. Some properties of $M$

Now $M$ is the class of those representing measure $m_\phi$ on the sphere $S$ which is the boundary of the open unit $B$ in $C^n$ that satisfies

$$\phi(f)=\int_S f\ dm_\phi$$

for every $f$ in $A(B)$. When $n=1$, $M$ has exactly one member, namely normalized Lebesgue measure on the unit circle.

LEMMA 2.1. $M$ is a convex set and weak -compact.

LEMMA 2.2. $M$ also has the corresponding weak*-topology. In general, it turns out to be a very large set when $n>1$. Moreover the members of $M$ are the circular probability measure $m_\theta$ on $S$, these satisfy

$$\int_S v(e^{i\theta}\zeta)\quad dm_\theta(\zeta)=\int_S v\ dm_\theta$$

for every $v$ in the class of continuous function $C(S)$ on $S$ for every real $\theta$.

To see some others, take $n=2$, for simplicity. Let $m_\tau$ be any probability measure on $\bar{U}(U$: open ball in $C^2)$, then the following fact is satisfied.

PROPOSITION 2.3. For every $g$ in $A(U)$,

$$\int_{\bar{U}} g\ \mathrm{dm}_\tau=\phi(g).$$

For example, $m_\tau$ might be concentrated on a simple closed curve in $U$ that surrounds the origin, in such a way that $m_\tau$ solves the Dirichlet problem at 0 relative to the domain bounded by a simple closed curve in $U$. Then the measure $m$ satisfies;

THEOREM 2.4. If the measure $m_\tau$ satisfies the following equation

$$\int_S v\, dm_\phi = \int_{\bar{U}} dm_\tau(z)\frac{1}{2\pi}\cdot\int_0^{2\pi} v(z, e^{i\theta}\sqrt{1-|z|^2})\, d\theta,$$

then $m_\tau$ belongs to $M$ for every $v \in C(S)$ and $z \in \bar{U}$.

PROOF. To see this, simply note that the inner integral on the right side of the theorem 2.4, with $v$ replaced by $f$ in $A(B)$, equals $f(z,0)$. The support of this $m_\tau$ is the set of all $(z,w)$ in $S$ for which $z$ lies in the support of $m_\tau$.

Furthermore the set $M$ plays a role in the study of the Lumer's Hardy space $(LH)^p(B)$ on the open unit ball $B$. First we introduce the definition of this space.

DEFINITION. The Lumer's Hardy space $(LH)^p(B)$ is the class of holomorphic in $B$ provided that $|f|^p$ has a pluri-harmonic majorant in $B$, i.e., provided that $|f|^p \leq Re\ g$ for some holomorphic $g$ in $B$ for $0 < p < \infty$.

We now list some consequences of the $(LH)^p(B)$ space. Since $(LH)^p(B)$ contains a closed subspace that is isomorphic to $l^\infty$ and lies in $H^\infty(B)$, it follows that $A(B)$ is separable in the norm topology of $(LH)^p(B)$.

PROPOSITION 2.5. (i) $(LH)^p(B)$ is not separable and $A(B)$ is dense in $(LH)^p(B)$

(ii) $(LH)^2(B)$ is not isomorphic to a Hilbert space.

To see the connection between $M$ and $(LH)^p$, associate to every continuous real function $v$ on $S$, the numbers;

$\alpha(v)=\sup\{\int_S v\, dm_\phi$: $v$:real continuous on $S$, $m\in M\}$

$\beta(v)=\inf\{u(0)$: $u\in Re\ A(B)$, $u\geq v$ on $S\}$.

Since every $m_\phi$ in $M$ satisfy $\phi(f)=\int_S f\, dm_\phi$ with $Re\, f$ in place of $f$, it is clear that $\alpha(v)\leq\beta(v)$. The converse inequality is proved in [7]. So we have the remark as following:

REMARK 2.6. The preceding numbers $\alpha(v)$, $\beta(v)$ are equal on $S$.

This remark and above facts imply the following.

THEOREM 2.7. A holomorphic $f$ in $D$ lies in $(LH)^p(B)$ if and only if

$$\sup_{r,m}\int_S |f(r)|^p\, dm_\phi<\infty$$

where $0<r<1$, $m_\phi\in M$.

## 3. Representing measures on the Silov boundary

Our discussion of maximal subalgebras of $C(X)$ has not involved any detailed information about the relation of the compact Hausdorff space $X$ to the algebra $A$. Further discussion requires the introduction of the maximal ideal space and Silov boundary for $A$.

Let $A$ be a closed subalegbra of $C(X)$, as usual containing the constants and separating points. The space of maximal ideals of $A$ is the set $M(A)$ of all non-zero complex linear functionals on $A$ which are multiplicative. Each such multiplicative functionals is automatically of norm 1 and we give to $M(A)$ the weak topology which it inherits as a subset of the unit shpere in the conjugate space of $A$. The space $M(A)$ is the largest compact Hausdorff space on which the algebra $A$ can be realized as a separating algebra of continuous functions.

In $M(A)$ there is a unique minimal closed subset $\Gamma(A)$ on which every *function in* $A$ *assumes its maximum* modulus. We call $\Gamma(A)$ the Silov boundary for $A$[8]. Since each function in $A$ assumes its maximum on $\Gamma(A)$ we may regard $A$ as a subalgebra of $C(\Gamma)$. The minimality of $\Gamma(A)$ we may regard $A$ as a of $C(\Gamma)$. The minimality of $\Gamma$ shows that $\Gamma$ is the smallest compact Hausdorff space on which $A$ can be realized as a closed separating algebra of continuous functions. So we can define the representing measure $m_\phi$ on the Silov boundary $\Gamma$ like the preceding methods as following. If $z\in M(A)$, there is a positive measure $m_\phi$ on $\Gamma$ such that

$$f(z)=\int_\Gamma f\ dm_\phi$$

for every $f$ in $A$. This representing results from the fact that any continuous linear functionals on $C(\Gamma)$ which has norm 1. To apply to the representing measure, let us make the following definition.

Definition. The algebra $A$ is called pervasive if $A$ is a pervasive subalgebra of $C(\Gamma)$.

It follows that if $A$ is a pervasive subalgebra of $C(X)$ then $X=\Gamma$ but $A$ may be pervasive on $\Gamma$ and not on $X$.

Theorem 3.1. Let $A$ be a pervasive subalgebra of $C(\Gamma)$, let $z\in M(A)-\Gamma$ and $m_\phi$ be any representing measure on $\Gamma$. Then the closed support of $m_\phi$ is all of $\Gamma$.

PROOF. Let $K$ be the closed support of $m_\phi$. Suppose $K$ is a proper closed subset of $\Gamma$. Since $f(z)=\int_K f\, dm_\phi$,

$$|f(z)|\leq \sup_K |f|,$$

and since $A$ is pervasive the measure $m_\phi$ defines a multiplicative linear functional on $C(K)$. Thus $m_\phi$ must be a point mass, which is absurd since $z\notin\Gamma$.

COROLLARY 3.2. Let $A$ be a pervasive subalgebra of $C(\Gamma)$ and let $f$ be a function in $A$ which has norm 1. If there is a point $z\in M(A)-\Gamma$ such that $|f(z)|=1$, then $f$ is a constant.

PROOF. Choose a representing measure $m_\phi$. Since $m_\phi$ has mass 1, $|f|\leq 1$, and

$$1=|f(z)|=|\int_\Gamma f\, dm_\phi|,$$

it is clear that $f(x)=f(z)$ for all $x$ in $\Gamma$.

*Of course Theorem 3.1 and its corollary* hold for essential maximal algebra. We have stated them for pervasive algebra to emphasize once again that the pervasive property of maximal algebras is the fundamental one.

PROPOSITION 3.3. Let $f$ be a function in $A$ which has norm 1, and let $K$ be the subset of $M(A)$ on which $f=1$. Let $A_K$ be the algebra obtained by restricting $A$ to the set $K$. Then $A_K$ is closed and

i) $M(A_K)=K$

ii) If $z\in K$, then any representing measure $m_\phi$ is support on $K\cap\Gamma$.

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