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ON SEMI-CLOSURE STRUCTURES AND TOPOLOGICAL MODIFYING STRUCTURES

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1. Semi-closure structures

Let X be any non empty set and $\mathscr{P}(X)$ the power set of X. A function $u:\mathscr{P}(X)\to\mathscr{P}(X)$ is called a semi-closure structure [3] on X if satisfies the following four conditions;

1) $u(\phi)=\phi$,
2) $A\subset u(A)$ for each $A\in\mathscr{P}(X)$.
3) $A\subset B \Rightarrow u(A)\subset u(B)$ for each $A,B\in\mathscr{P}(X)$, and
4) $u(A)=u(u(A))$ for each $A\in\mathscr{P}(X)$.

A pair (X,u) where u is a semi-closure structure on X, is called a semi-closure space. These concepts are generalizations of the more familiar Kuratowski closure operator and topological spaces, respectively. For a convinience, we shall agree to use $\mathscr{U}$ as $\{A\subset X|\ u(X-A)=X-A\}$. Clearly, a semi-closure structure u is satisfied i) $X,\phi\in\mathscr{U}$ and ii) for every $A\in\mathscr{U}$ $i\in I$, $\bigcup_{i\in I}A\in\mathscr{U}$, but the finite intersection of elements of $\mathscr{U}$ is not an element of $\mathscr{U}$, in general (A family $\mathscr{U}$ of subsets of X satisfying the above conditions i) and ii) is called a pretopology [1], a supratopology [2], or a semi-topology [6] for X.).

The concept of semi-closure structures is motivated by the following examples;

EXAMPLE 1.1. Let $(X, \mathscr{T})$ be a topological space and $-$ and 0 denote the closure operator and the interior operator in X, respectively. Then

$$\mathscr{T}^{\alpha}=\{A\subset X \mid A\subset A^{0-}\},$$
$$\mathscr{T}^{\beta}=\{A\subset X \mid A\subset A^{-0}\}, \text{ and}$$
$$\mathscr{T}^{\gamma}=\{A\subset X \mid A\subset A^{-0-}\}$$

are pretopologies but not topologies for X[1,5].

EXAMPLE 1.2[6]. Let X and Y be any two non empty sets and $\mathscr{F}$ a subcollection of $\{f \mid f: X\to Y$ is a function$\}$. Let $K(f,g)$ denote the coincidence set of f and g, consisting of all points $x\in X$ such that $f(x)=g(x)$. Define $u:\mathscr{P}(X)\to\mathscr{P}(X)$ by

$$u(A)=\bigcap\{K(f,g) \mid K(f,g)\supset A, f,g\in\mathscr{F}\}.$$

Then if $\bigcap_{f,g\in F} K(f,g)=\phi$, then u is a semi-closure structure on X.

Moreover, if $Y=\{0,1\}$, then the above semi-closure stucture u is a Kuratowski closure operator.

EXAMPLE 1.3 [8]. Let X be any non empty set and G and $\mathscr{E}$ denote a transformation group of X and the equivalence relations of X, respectively. Then between the complete lattice $\mathscr{G}$ (the set of all subgroups of G) and the complete lattice $\mathscr{E}$ there can be established a dual (inverse) Galois connection[7] $\mathscr{G}\underset{\tau}{\overset{\sigma}{\rightleftarrows}}\mathscr{E}$ such that

1) $\sigma(A)=\{a\sim b \mid f(a)=b$, for some $f\in A\}$ for each subgroup A of G and

2) $\tau(\sim)=\{f\in G \mid f(x)\sim x$, for any $x\in X\}$ for each $\sim\in\mathscr{E}$.

By the Galois connection $(\sigma\tau, \tau\sigma)$, we can prove that if

$\sigma(\phi)=\phi$ and $\tau(\phi)=\phi$, then $\sigma\tau$ and $\tau\sigma$ are semi-closure structures on $\mathscr{G}$ and $\mathscr{E}$, respectively. In [7], these structures $\sigma\tau$ and $\tau\sigma$ are called closure operators.

2. Topological modifying structures

Let X be a non empty set and let Γ be a collection of semi-closure structures on X[3, 4]. Γ is called a topological modifying structure on X if for each $A, B \in \mathscr{P}(X)$ and for each $u, v \in \Gamma$, there exists an element w in Γ such that $u(A)\cup v(B)\supset w(A\cup B)$. Let (X, u) be a semi-closure space. We let $\Phi_u(x)=\{A\subset X\colon x \notin u(A^c)\}$. In a topological space (X, u), $\Phi_u(x)$ is clearly the neighborhood system at x in (X, u) for each $x \in X$.

REMARK. (1) If a topological modifying structure Γ on X has only one element u, then u satisfies the Kuratowski closure axioms. From now on, we shall agree to use u as the unigue topology for X determined by u.

(2) Any collection of semi-closure structures on X is not a topological modifying structure on X, in general, as shown by the following example A.

(3) Any collection of topologies for X which has at least two elements is not a topological modifying structure on X, in general, as shown by the following example B.

EXAMPLE A. Let $(X, \mathscr{T})$ be a topological space and and denote the closure operator and the interior operator in X, respectively. Then

$$\mathscr{T}^{\alpha}=\{A\subset X\colon A\subset A^{0-0}\},$$
$$\mathscr{T}^{\beta}=\{A\subset X\colon A\subset A^{-0}\} \text{ and}$$
$$\mathscr{T}^{\gamma}=\{A\subset X\colon A\subset A^{-0-}\}$$

are pretopologies [1,5], but not topologies for X. If u, v, and w are semi-closure structures on X determined by $\mathscr{T}^{\alpha}$, $\mathscr{T}^{\beta}$, and $\mathscr{T}^{\gamma}$, respectively, then $\Gamma=\{u,v,w\}$ is not a topological modifying structure on X.

EXAMPLE B. Let $X=\{a,b,c\}$ and $u=\{X,\phi,\{a\},\{a,b\}\}$ and $v=\{X,\phi,\{b\},\{b,c\}\}$. Then u and v are topologies for X and $u\{c\}\cup v\{a\}=\{a,c\}\not\supset u\{a,c\}=v\{a,c\}=\{a,b,c\}$. Therefore $\Gamma=\{u,v\}$ is not a topological modifying structure on X.

THEOREM 2.1. Let Γ be a topological modifying struc ture on a set X. Then $\bigcup_{u\in\Gamma}\Phi_u(x)$ is a neighborhood system at x, for each $x\in X$. That is, Γ determines a topology T_Γ for X.

PROOF. 1) Set $N_x=\bigcup_{u\in\Gamma}\Phi_u(x)$ and let $A\in N_x$. Then there is $u\in\Gamma$ such that $x\notin u(A^c)$. Since $A^c\subset u(A^c)$, $x\notin A^c$ and thus $x\in A$.

2) Let A and B be two elements of N_x. Then there are $u,v\in\Gamma$ such that $x\notin u(A^c)$ and $x\notin v(B^c)$. Since Γ is a topological modifying structure on X, there exists $w\in\Gamma$ such that $x\notin u(A^c)\cup v(B^c)\supset w(A^c\cup B^c)=w((A\cap B)^c)$. Now we have $x\notin w((A\cap B)^c)$ and thus $A\cap B\in N_x$.

3) Let $A\in N_x$ and $A\subset B\subset X$. Then there exists $u\in\Gamma$ such that $x\notin u(A^c)$. Since $A\subset B$, $B^c\subset A^c$ and $u(B^c)\subset u(A^c)$. It follows that $x\notin u(B^c)$ and thus $B\in N_x$.

4) Let $A\in N_x$. Then there exists $u\in\Gamma$ such that $x\notin u(A^c)$ and we have $x\in X-u(A^c)\subset A$. Let $B=X-u(A^c)$. Then we shall prove that i) $B\in N_x$ and ii) $A\in N_y$, for each $y\in B$, that is, for each $y\in B$, $y\notin w(A^c)$ for some $w\in\Gamma$.

i) Since u is a semi-closure structure on $X(i.e., u \varepsilon \Gamma)$, $u(B^c)=u((X-u(A^c))^c)=u(u(A^c))=u(A^c)$. Since $x \notin u(A^c)$, $x \notin u((X-u(A^c))^c)$ and thus $B=X-u(A^c) \varepsilon N_x$.

ii) Since $B \cap u\ (A^c)=\phi$, $y \notin u(A^c)$ and thus $A \varepsilon N$, for each $y \varepsilon B$.

The proof is complete.

THEOREM 2.2. Let $(X, \mathscr{T})$ be a topological space and let Γ be a collection of semi-closure structures on X such that $\mathscr{T} \varepsilon \Gamma$ and for each $u \varepsilon \Gamma, u \subset \mathscr{T}$. Then,

(1) Γ is a topological modifying structure on X.

(2) $\mathscr{T}_\Gamma = \mathscr{T}$.

PROOF. (1) Let u be the closure structure on $(X, \mathscr{T})$. For each $v, w \varepsilon \Gamma$ and for each $A, B \varepsilon \mathscr{P}(X)$,

$$u(A) \cup w(B) \supset u(A) \cup u(B) = u(A \cup B).$$

Thus Γ is a topological modifying structure on X.

(2) Let A be a neighborhood of x in $(X, \mathscr{T}_\Gamma)$. Then there exists an element v in Γ such that $x \notin v\ (A^c)$. Since $v\ (A) \supset u(B)$ for each $A \varepsilon \mathscr{P}(X)$, $x \notin u(A^c)$. Thus A is a neighborhood of x in $(X, \mathscr{T})$. Conversely, let A be a neighborhood of x in $(X, \mathscr{T})$. Then $x \notin u(A^c)$ and thus $A \ \varepsilon \ \Phi_u(x) \subset \bigcup_{v \varepsilon \Gamma} \Phi_v(x)$. Therefore A is a neighborhood of x in $(X, \mathscr{T}_\Gamma)$.

The proof is complete.

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