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EVOLUTION EQUATION OF TYPE $\frac{du}{dt}+A\beta u \ni 0$ IN $L^{\infty}(\Omega)$

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1. Introduction.

Let Ω be a measure space of bounded measure. For $1\leq p\leq+\infty$, $L^p(\Omega)$ denotes the Lebesgue space of Ω with norm $\|\cdot\|_p$. Let A be an operator of $L^p(\Omega)$ and $\beta:\Omega\times R\to\mathscr{P}(R)$ a mapping. Define the operator $A\beta$ of $L^p(\Omega)$ by

$$A\beta=\{[u,w]\varepsilon L^p(\Omega)\times L^p(\Omega)|\text{there exists } v\varepsilon L^p(\Omega) \text{ such that } [v,w]\varepsilon A \text{ and } a.e.\, x\varepsilon\Omega,\ v(x)\varepsilon\beta(x,u(x))\}.$$

The purpose of this paper is to study the Cauchy problem

$$\frac{du}{dt}+A\beta u \ni 0,\ \ u(0)=u_0$$

in $L^{\infty}(\Omega)$.

Suppose that the following conditions are satisfied:

(H1) A is an m-accretive operator of $L^{\infty}(\Omega)$ and monotone in $L^2(\Omega)$,

(H2) $a.e.\, x\varepsilon\Omega$, $r\varepsilon R\to\beta(x,r)\varepsilon\mathscr{P}(R)$ is maximal monotone in R,

(H3) for every $r\varepsilon R$, there exists $v\varepsilon L^{\infty}(\Omega)$ such that $a.e.$ $x\varepsilon\Omega$, $v(x)\varepsilon\ \beta(x,r)$,

(H4) for every $f\varepsilon L^{\infty}(\Omega)$ and $\lambda>0$, there exists at most one solution $u\varepsilon L^{\infty}(\Omega)$ of $u+\lambda A\beta u\ni f$.

THEOREM 1. ([5])

Let (H1)-(H4) be satisfied. Suppose $A\beta\neq\phi$. Then $A\beta$ is m-accretive in $L^{\infty}(\Omega)$.

2. Solutions of $\frac{du}{dt}+A\beta u \ni 0$ in $L^\infty(\Omega)$.

By [1] and Theorem 1, we have the following theorem on the integral solution in the sense of Bénilan of the equation:

THEOREM 2. Let (H1)-(H4) be satisfied. Suppose $A\beta \neq \Phi$. Let $g \in L^1(0,T;L^\infty(\Omega))$ for $T>0$. Then for every $u_0 \in \overline{D(A\beta)}$, there exists a unique integral solution u on $[0, T]$ of $du/dt+A\beta u \ni g$, $u(0)=u_0$. Moreover $u(t) \in \overline{D(A\beta)}$ for every $t \in [0,T]$.

Suppose the following condition is satisfied:
(HC) A is m-accretive in $L^\infty(\Omega)$ and cyclically monotone in $L^2(\Omega)$ ([2]).

REMARK 3. ([4]).
Let (HC) be satisfied. Then (H1) is satisfied and there exists a proper lower semi-continuous convex function ϕ defined on $L^2(\Omega)$ into $(-\infty,+\infty]$ with $\partial\phi=A_2$, where A_2 is the closure of A in $L^2(\Omega)$.

THEOREM 4.
Let (HC), (H2)—(H4) be satisfied. Then for every $u_0 \in D(A\beta)$, there exists $u \in \mathscr{C}([0,T];L^\infty(\Omega))$ such that

$$\begin{cases} \frac{du}{dt}(t) \in L^\infty(0,\mathrm{T};\ \sigma(L^\infty(\Omega),L^1((\Omega)))) \\ \frac{du}{dt}+A\beta u(t) \ni 0 \text{ a.e. on } (0,T) \\ u(0)=u_0 \end{cases}$$

for $T>0$.

PROOF. By Theorem 1, $A\beta$ is m-accretive in $L^\infty(\Omega)$. Consider the approximation equation

$$(1)\qquad \begin{cases} \dfrac{u(t)-u_\varepsilon(t-\varepsilon)}{\varepsilon}+A\beta u_\varepsilon(t)\ni 0 & (t>0) \\ u(t)=u_0 & (t\leq 0) \end{cases}$$

for every $\varepsilon>0$ in $L^\infty(\Omega)$. The equation (1) is written by

$$\begin{cases} u_\varepsilon(t)=(I+\varepsilon AB)^{-1}u_\varepsilon(t-\varepsilon) & (t>0) \\ u_\varepsilon(t)=u_0 & (t\leq 0) \end{cases}$$

and it has at most one solution

$$u_\varepsilon(t)=(I+\varepsilon A\beta)^{-1}u_0 \text{ for every } t\varepsilon((k-1)\varepsilon, k\varepsilon]$$

or

$$u_\varepsilon(t)=(I+\varepsilon A\beta)^{-(t/\varepsilon)}u_0 \text{ for every } t>0.$$

By [3], $u_\varepsilon(t)$ converges to $u(t)$ in $L^\infty(\Omega)$ uniformly on every compact interval of $[0,\infty)$ as $\varepsilon\to 0+$ and if we put $u(t)=S(t)u_0$, then $S(t)$ is a nonlinear semigroup of contraction generated by $A\beta$. By [1], the limit $u(t)$ is the integral solution on $(0, \infty)$ of $\dfrac{du}{dt}+A\beta u\in 0$, $u(0)=u_0$ and $\|u(t)-u(s)\|_\infty\leq|t-s|\|A\beta u_0\|_\infty$ for every $t, s\geq 0$. Thus $a.e.$ $t\varepsilon(0,\infty)$, $u(t)$ is weakly differentiable in $L^\infty(\Omega)$. We put

$$w_\varepsilon(t)=\frac{u_\varepsilon(t)-u_\varepsilon(t-\varepsilon)}{\varepsilon}.$$

For every $t\geq 0$,

$$(2)\quad \|w_\varepsilon(t)\|_\infty=\frac{1}{\varepsilon}\|(I+\varepsilon A\beta)^{-(t/\varepsilon)}u_0-(I+\varepsilon A\beta)^{-(t/\varepsilon)+1}u_0\|_\infty \\ \leq\|A\beta u_0\|_\infty.$$

Hence w_ε is bounded in $L^\infty((0,\infty)\times\Omega)$. Let $v_\varepsilon(t)\varepsilon L^\infty(\Omega)$ such that

$$(3)\qquad w_\varepsilon(t)+Av_\varepsilon(t)\ni 0$$

and $a.e. x\varepsilon\Omega$, $v_\varepsilon(t,x)\varepsilon\beta(x,u_\varepsilon(t,x))$. Since u_ε is bounded in

$L^\infty((0,\infty)\times\Omega)$, by (H3), v_ε is bounded in $L^\infty((0,\infty)\times\Omega)$. There exists $\{\varepsilon_n\}$, $\varepsilon_n>0$, $\varepsilon_n\to 0+$ such that $v_{\varepsilon_n}\to v$ and $w_{\varepsilon_n}\to w$ in $L^\infty((0,\infty)\times\Omega)$. As in the proof of Theorem II.8 in [4], we have

$$\frac{du}{dt}=w \text{ in } \mathscr{D}'(0,\infty;L^2(\Omega))$$

and by (2), $\|\frac{du}{dt}(t)\|_\infty\leq\|A\beta u_0\|_\infty$.

Let j: $\Omega\times R\to(-\infty,\infty]$ be a proper lower semi-continuous convex function such that $a.e.\ x\varepsilon\Omega$, $\partial j(x,.)=\beta(x,.)$. We define J: $L^2(\Omega)\to(-\infty,\infty]$ by

$$J(z)=\begin{cases}\int_\Omega j(z) & \text{if } j(z)\varepsilon L^1(\Omega)\\ +\infty & \text{otherwise}\end{cases}$$

for every $z\varepsilon L^2(\Omega)$. Then J is a proper lower semi-continuous convex function on $L^2(\Omega)$ into $(-\infty,\infty]$ and the subdifferential ∂J of J is the prolongation of β to $L^2(\Omega)$ ([2]). Since $u_{\varepsilon_n}\to u$, $v_{\varepsilon_n}\to v$ in $L^\infty((0,\infty)\times\Omega)$ and

(4) $$v_{\varepsilon_n}(t)\varepsilon\partial Ju_{\varepsilon_n}(t),$$

it follows that $u(t)\varepsilon D(\partial J)$ and $v(t)\ \varepsilon\partial Ju(t)$ $a.e.\ t\varepsilon(0,\infty)$. Thus

(5) $$v(t)\varepsilon\beta u(t)\ \ a.e.\ t\varepsilon(0,\infty).$$

By (4), we have

(6) $$\int_\Omega v_{\varepsilon_n}(u_{\varepsilon_n}(t)-u_{\varepsilon_n}(t-\varepsilon_n))\geq J(u_{\varepsilon_n}(t)-J(u_{\varepsilon_n}(t-\varepsilon_n)).$$

By Remark 3, there exists a proper lower semi-continuous convex function ϕ defind on $L^2(\Omega)$ into $(-\infty,\infty]$ with $\partial\phi=A_2$. Let $T>0$. We define Φ: $L^2((0,T)\times\Omega)\to(-\infty,\infty)$ by

$$\Phi(z)=\begin{cases}\int_0^T\int_\Omega\phi(z) & \text{if } \phi(z)\varepsilon L^1((0,T)\times\Omega)\\ +\infty & \text{otherwise}\end{cases}$$

for every $z \in L^2((0, T)\times\Omega)$. Then Φ is a proper lower semicontinuous convex function on $L^2((0, T)\times\Omega)$ into $(-\infty, \infty]$ and the subdifferential $\partial\Phi$ of Φ is the prolongation of $\partial\phi$ to $L^2((0, T)\times\Omega)$. By (3), $w_{\varepsilon_n}(t)+A_2 v_{\varepsilon_n}(t)\ni 0$ and thus $w_{\varepsilon_n}+\partial\Phi v_{\varepsilon_n}\ni 0$. For every $z\in L^2((0, T)\times\Omega)$,

$$\Phi(z)-\Phi(v_{\varepsilon_n})\geq\int_0^T\int_\Omega(-w_{\varepsilon_n})\ (z-v_{\varepsilon_n})$$
$$=\int_0^T\int_\Omega\ (-w_{\varepsilon_n})z+\int_0^T\int_\Omega w_{\varepsilon_n}v_{\varepsilon_n}.$$

By (6),

$$\int_0^T w_{\varepsilon_n}v_{\varepsilon_n}=\int_0^T \frac{u_{\varepsilon_n}(t)-u_{\varepsilon_n}(t-\varepsilon_n)}{\varepsilon_n}\,v_{\varepsilon_n}$$
$$=\int_0^T \frac{J(u_{\varepsilon_n}(t))-J(u_{\varepsilon_n}(t+\varepsilon_n))}{\varepsilon_n}$$
$$=\frac{1}{\varepsilon^n}\sum_{i=1}^{k}\int_{(i-1)\varepsilon_n}^{i\varepsilon_n}(J(u_{\varepsilon_n}(t))-J(u_{\varepsilon_n}(t-\varepsilon_n)))$$
$$=\frac{1}{\varepsilon_n}(\int_{(k-1)\varepsilon_n}^{k\varepsilon_n} J(u_{\varepsilon_n}\ (T))-\int_0^{\varepsilon_n} J(u_0))$$

(7) $$=J(u_{\varepsilon_n}(T))-J(u_0),$$

where $T=k\varepsilon_n$. By [2], $t\to J(u(t))$ is absolutely continuous and $a.e.\ t\in(0, T)$,

(8) $$\frac{d}{dt}J(u(t))=\int_\Omega v(t)\frac{du}{dt}(t).$$

By (8),

$$\int_0^T\int_\Omega v(t)\frac{du}{dt}\,(t)=\int_0^T\frac{d}{dt}(u(t))=J(u(T))-J(u_0)$$

(9) $$=J(u(T))-J(u_{\varepsilon_n}(T))+J(u_{\varepsilon_n}(T)-J(u_0).$$

By (7)-(9),

$$\Phi(z)\geq\Phi(v_{\varepsilon_n})+\int_0^T\int_\Omega(-w_{\varepsilon_n})z$$

$$+\int_0^T \int_\Omega v(t) \frac{du}{dt}(t)+J(u_{\varepsilon_n}(T))-J(u(T)).$$

Since $\Phi(v) \leq \lim_{\varepsilon_n \to 0+} \Phi(v_{\varepsilon_n})$ and $J(u(t)) \leq \lim_{\varepsilon_n \to 0+} J(u_{\varepsilon_n}(T))$,

$$\Phi(z) \geq \Phi(v)+\int_0^T \int_\Omega (-\frac{du}{dt})z+\int_0^T \int_\Omega v \frac{du}{dt}.$$

Hence for every $z \in L^2((0,T)\times\Omega)$,

$$\Phi(z)-\Phi(v) \geq \int_0^T \int_\Omega (-\frac{du}{dt}) \ (z-v).$$

Thus $-\frac{du}{dt} \varepsilon \partial\phi(v)$, that is $\frac{du}{dt}(t)+A_2 v(t) \ni 0$ $a.e. t \ \varepsilon(0, T)$. Since $v(t) \varepsilon L^\infty(\Omega)$, $\frac{du}{dt}(t)+Av(t) \ni 0$ $a.e. t \varepsilon (0, T)$. By (5), $\frac{du}{dt}(t)+A\beta u(t) \ni 0$ $a.e. t\varepsilon(0,T)$.

3. Example.

Let $\phi: L^2(\Omega)\to[0,\infty]$ be a function with $\phi(0)=0$ satisfying:

(10) $$\phi(u_1-p(u_1-p(u_1-u_2))+\phi(u_2+p(u_1-u_2)) \leq \phi(u_1)+\phi(u_2)$$

for every u_1, $u_2 \varepsilon L^2(\Omega)$ and $p \varepsilon \mathscr{C}^1(R)$ with $0<p'<1$, $p(0)=0$.

If $\phi: L^2(\Omega)\to[0,\infty]$ is a lower semi-continuous convex function with $\phi(0)=0$ satisfying (10), then $A=\partial\phi \cap L^\infty(\Omega)\times L^\infty(\Omega)$ satisfies (HC)([4]).

Let Ω be a bounded open subset of R^N with smooth bord Γ. Let $j: R\to[0,\infty]$ be a lower semi-continuous convex function with $j(0)=0$ and $\gamma=\partial j$. We define a function ϕ on $L^2(\Omega)$ by

$$(11)\qquad \phi(u)=\begin{cases}\int_\Omega \frac{1}{2}\,|\mathrm{grad} u|^2+\int_\Gamma j(u) & \text{if } u\in H^1(\Omega)\\ +\infty & \text{otherwise.}\end{cases}$$

Then $\phi: L^2(\Omega)\to[0,\infty]$ is a lower semi-continuous convex function with $\phi(0)=0$ and the subdifferential $\partial\phi$ of ϕ:

$$(12)\quad \partial\phi=\{(u,v)\in L^2(\Omega)\times L^2(\Omega)\,|\, u\in H^2(\Omega),\ v=-\Delta u \ a.e. \text{ on } \Omega \text{ and } \frac{\partial u}{\partial n}+\gamma(u)\ni 0 \ a.e. \text{ on } \Gamma\},$$

where $\frac{\partial u}{\partial n}$ is the exterior normal derivative([4]).

By [4], ϕ of (11) satisfies (10). Thus $A=\partial\phi\cap L^\infty(\Omega)\times L^\infty(\Omega)$ for $\partial\phi$ of (12) satisfies (HC) ([4]).

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References

[1] Ph. Bénilan, Equation d'évolution dans un espace de Banach quelconque et applications, Thèse, Université de Paris XI, 1972.

[2] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, Mathe. Studies 5, North Holland, 1973.

[3] M. G. Crandall and T. M. Liggett, Generation of nonlinear transformations on general Banach spaces, Amer. Jour. Math. 93(1971), 265-298.

[4] K. S. Ha, Sur des semi-groupes non linéaires dans les espaces

de $L^\infty(\Omega)$, Jour. Math. Soc. Jap. 31(1979), 593-622.

[5] K. S. Ha, Multiplicative perturbation of m-accretive operateors and evolution equations in $L^\infty(\Omega)$, Lect. Not. Pur. Appl. Math. 90(1984), 221-226.

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