PUSAN KYŎNGNAM MATHEMATICAL JOURNAL
Vol. 1, 21~24, 1985

ON SEMIGROUP RINGS(I)

TAE YOUNG HUR AND CHOL ON KIM

In this paper K will be a field, S-a semigroup ring of the semigroup S with coefficients in K will be denoted by $K[S]$ ($c.f.$ Clifford[2]).

We shall prove that if $K[S]$ is right Artinian with a cancellative semigroup S, then S must be finite. Thus, we get a result related to the Woods' Theorem[5] for semigroup ring case.

We recall that a semigroup S will be called cancellative if $G=SS^{-1}$ be a group and for all $s,t\varepsilon S$, there exist $s_1,t_1 \varepsilon S$ such that $st^{-1}=t_1s_1^{-1}$.

We begin with the following well-known result which is a kind of common denominator theorem.

LEMMA 1. Let S be a cancellative semigroup and $G=SS^{-1}$ be a group. If $s_1,s_2,\cdots, s_n\varepsilon S$, then there exist $r_1,r_2,\cdots, r_n\varepsilon S$ and $s\varepsilon S$ such that $s_i^{-1}=r_is^{-1}$ for $1\leq i\leq n$.

PROOF. The proof will be by induction on n. For $n=1$, take $s=s_1^2$ and $r_1=s_1$. Assuming the result to be ture for all positive integers less than n. By induction hypothesis, there exist $r_1^*,r_2^*, \cdots,r_{n-1}^* \varepsilon S$ and $s^* \varepsilon S$ such that

$$s_i^{-1}=r_i^*s^*$$

for $1\leq i\leq n-1$. Since S is cancellative, there exist r_n, $r\varepsilon S$ such that $s_nr_n=s^*\ r=s$ say. Hence $s_n^{-1}=r\ s^{-1}$. Set $r_i=r_i{}^*r$ for $1\leq i\leq n-1$ so that $s_i^{-1}=r_i^*\,s^{*-1}=(r_i^*\,r)(r^{-1}s^{*-1})$

$$=(r_i^* r)(s^*r)^{-1}=r_i s^{-1}$$

for $1\leq i\leq n-1$. This completes the induction and concludes the proof.

Okninski[4] showed that if a semigroup ring $K[S]$ is left and right Artinian, then S is finite. In order to consider the above theorem to the semigroup ring case with one sided Artinian, we start with the following.

LEMMA 2. Let K be a field and S- cancellative semigroup. If I is a right ideal of group ring $K[G]$, then $I\cap K[S]$ is a right ideal of $K[S]$ and $(I\cap K[S])\ K[G]=I$.

PROOF. It is easy to see that $I\cap K[S]$ is a right ideal of $K[S]$. Also we get $(I\cap K[S])K[G]\subseteq I\ K[G]\subseteq I$ because of $I\cap K[S]\subseteq I$.

On the other hand, let $\alpha=a_1g_1+a_2g_2+\cdots+a_ng_n$ be an element of I. Since $G=SS^{-1}$, there exist $s_i, t_i \varepsilon S$ such that $g_i=s_it_i^{-1}$ for $1\leq i\leq n$. Thus $\alpha=a_1s_1t_1^{-1}+a_2s_2t_2^{-1}+\cdots+a_ns_nt_n^{-1}$.

From lemma 1, we get $t_i^{-1}=s_i'\ t^{-1}$ for $1\leq i\leq n$. Set $s_is_i'=\alpha_i \varepsilon S$. Then we have $\alpha=(a_1s_1s_1'+\cdots+a_ns_ns_n')t^{-1}=(a_1\alpha_1+\cdots+a_n\alpha_n)t^{-1}$ and $\alpha t=a_1\alpha_1+\cdots+a_n\alpha_n\varepsilon\ I$ because of $\alpha\varepsilon\ I$. Since $(a_1\alpha_1+\cdots+a_n\alpha_n)\ \varepsilon I\cap K[S]$, $\alpha=(a_1\alpha_1+\cdots+a_n\alpha_n)t^{-1}$ $\varepsilon(I\cap K[S])K[G]$. Thus $I\subseteq(I\cap K[S])K[G]$.

Now we are in a position to prove one of our main results.

THEEOREM 3. Let K be a field and S a cancellative semigroup. If semigroup ring $K[S]$ is right Artinian, then S is finite.

PROOF. Let $I_1\supseteq I_2\supseteq\cdots$ be a descending chain of right ideals of $K[G]$. Then $I_1\cap K[S]\supseteq I_2\cap K[S]\supseteq\cdots$ is also descending chain of right ideals of $K[S]$. Since $K[S]$ satisfies descending chain condition on right ideals of $K[S]$, there is an integer N such that $I_k\cap K[S]=I_N$ for all $k\geq N$. Since $(I_k\cap K[S])K[G]=I_k$, $I_k=I_N$ for all $k\geq N$. It means that $K[G]$ is right Artinian. By Connell[3, Theorem 3.1], G is finite. Hence S is also finite.

The ring R is called right perfect if R is semilocal (*i.e.* $R/J(R)$ is Artinian) and the Jacobson radical $J(R)$ is right T-nilpotent, or equivalently, R satisfies the descending chain condition on principal left ideals [1, Theorem 28.4, p. 315].

S. M. Woods [4] has shown that the groupring $R[G]$ is right(left) perfect if and only if R is right(left) perfect and G is finite.

Now let us consider the perfect semigroup ring case.

THEOREM 4. Let S be commutative cancellative semigroup. If $K[S]$ is left perfect, then S is finite.

PROOF. By Woods' result, we must show that $K[G]$ is left perfect. Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be elements of $K[S]$ and $\alpha_1K[G]\supseteq\alpha_2K[G]\supseteq\alpha_3K[G]\supseteq\cdots$ be a descending chain of principal right ideals of $K[G]$. By lemma 1, $\alpha_i=\beta_is_i^{-1}$ for some $\beta_i\varepsilon K[S]$ and $s\varepsilon S$. Thus we get $\beta_1s_1^{-1}K[G]\supseteq\beta_2s_2^{-1}K[G]\supseteq\cdots$ and $\beta_1K[G]\supseteq\beta_2K[G]\supseteq\cdots$ because of $S\subset K[G]$. Since $\beta_2=\beta_2\cdot 1\varepsilon\beta_2K[G]\subseteq\beta_1K[G]$, $\beta_2=\beta_1\gamma$ for some $\gamma\varepsilon K[G]$. Hence $\gamma=\gamma_0t^{-1}$ for some $\gamma_0\varepsilon K[S]$ and $t\varepsilon S$. Thus, we have $\beta_2=\beta_1\gamma_0t^{-1}$ and $t\beta_2=\beta_2t=\beta_1\gamma_0\varepsilon\beta_1K[S]$. Finally we get $t\beta_2K[S]\subseteq\beta_1K[S]$.

By the same reason $\beta_2tK[S]\supseteq t_1t\beta_3K[S]$ for some $t_1\varepsilon S$. Continuing in this fashion, we get the descending chain $\beta_1 K[S] \supseteq \tau_2\beta_2 K[S] \supseteq \tau_3\beta_3 K[S] \supseteq \cdots$ of principal right ideals of $K[S]$. Since $K[S]$ is left perfect, there exists an integer N such that $\tau_k\beta_k K[S] = \tau_N\beta_N K[S]$ for all $k \geq N$. Since S is commutative, $\beta_k\tau_k K[S] = \beta_N\tau_N K[S]$ for all $k \geq N$. Thus $\beta_k\tau_k K[S]K[G] = \beta_N\tau_N K[S]K[G]$ for all $k \geq N$ and $\beta_k\tau_k K[G] = \beta_N\tau_N K[G]$ for all $k \geq N$. So for all $k \geq N$, $\beta_k K[G] = \beta_N K[G]$ and $\beta_k s_k^{-1} K[G] = \beta_N s_N^{-1} K[G]$, we get $\alpha_k K[G] = \alpha_N K[G]$ for all $k \geq N$. Therefore $K[G]$ satisfies the descending chain condition on the principal right ideals of $K[G]$. Hence $K[G]$ is left perfect and the proof is complete.

Corollary. If K is any field, then $K[x]$ is not perfect.

PROOF. We have known that $S = \{x^i \mid i = 0, 1, 2, \cdots\}$ is a cancellative semigroup and $K[S]$ is just the polynomial ring $K[x]$ of x over K. So by the theorem $K[x]$ never perfect since S is finite.

References

[1] F. Anderson and K. R. Fuller, Rings and categories of modules, Springer-Verlag, Berlin, Heidelberg and New York, 1974.

[2] A. H. Clifford and G. B. Preston, The algebraic theory of semigroup, Providence, 1961.

[3] I. G. Connell, On the group ring, Canad., J. Math. 15(1963) 650-685.

[4] J. Okninski, Artinian semigroup rings, Communication in Algebra, 10(1982), 109-114.

[5] S. M. Woods, On perfect group rings, Pro. Amer. Math. Soc. 27(1971). 49-52.

Pusan National University