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ON THE NUMERICAL RANGES AND LUMER'S FORMULA

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1. Introduction

In [4], Kim and Yang defined a numerical range for the class of all numerically bounded (nonlinear) maps on a Hilbert C*-module, and gave some of the basic properties of such numerical range. In this paper which is a continuation of [4], we define the numerical range for the numerically bounded vector fields on the unit sphere of a Hilbert C*-module, and give additional properties of such numerical ranges. In particular we obtain an analogue of Lumer's formula for the class of Lipschitz maps.

Throughout this paper we let B be a unital C*-algebra, B^* its dual space, and X the Hilbert B-module with a B-valued inner product $\langle , \rangle$[5]. A Hilbert B-module X is assumed to have a vector space structure over the complex numbers C compatible with that of B in the sense that

$$\lambda(xb)=(\lambda x)b=x(\lambda b) \quad (x\varepsilon X,\ b\varepsilon B,\ \lambda\varepsilon \mathbf{C}).$$

We define the norm $\|\cdot\|_X$ on X by $\|x\|_X=\|\langle x,x\rangle\|^{\frac{1}{2}}$. We will use the following notations. $Q^*(X)$ is the vector space of all B^*-quasibounded maps. $W^*(X)$ is the vector space of all B*-numerically bounded maps. $L(X)$ is the Banach space of all bounded linear operators on X. we also denote the operator norm on $L(X)$ by $\|\cdot\|$.

2. Numerical range for nonlinear operators

If we set

$$\Pi_r = \{(x, f) \varepsilon X \times B^* : \|x\|_X = \|f\| \geqq r,\ f(\langle x, x \rangle) = \|x\|_X^2\} \quad (r > 0)$$

and $\Pi_0 = \bigcup_{r>0} \Pi_r$, then each $\Pi_r (r>0)$ and Π_0 are connected subsets of $X \times B^*$ with the norm$\times$weak* topology, unless X has dimension one over R[4]. From now on we shall assume that Π_0 has the norm$\times$weak* topology as a subset of $X \times B^*$. Also we shall assume that X doesn't have dimension one over R.

PROPOSITION 2.1. Let F: $\Pi_0 \to X$ be a continuous map such that $\|F(x,f)\|_X = \|x\|_X$ for $(x,f) \varepsilon \Pi_0$. Then $z \varepsilon \Sigma^*(F)$ implies $|z| = 1$, where $\Sigma^*(F)$ denotes the B*-asymptotic spectrum of $F \varepsilon Q^*(X)$[4].

PROOF. Let $z \varepsilon \Sigma^*(F)$. Then by definition of $\Sigma^*(F)$ we can find $(x_n, f_n) \varepsilon \Pi_0$ such that $\|x_n\|_X \geqq n$ and

$$\|(z\pi - F)(x_n, f_n)\|_X \leqq \frac{1}{n} \|x_n\|_X,$$

where π denotes the natural projection of $X \times B^*$ onto X.

$$\text{Hence } \|F(x_n, f_n)\|_X - \frac{1}{n}\|x_n\|_X \leqq |z| \|x_n\|_X \leqq \|F(x_n, f_n)\|_X + \frac{1}{n}\|x_n\|_X.$$

Using the assumption on F,

$$(1 - \frac{1}{n})\|x_n\|_X \leqq |z| \|x_n\|_X \leqq (1 + \frac{1}{n})\|x_n\|_X.$$

Dividing by $\|x_n\|_X$ and letting $n \to \infty$ completes the proof.

We note that the B^*-numerical range $\Omega^*(F)$ of $F \varepsilon W^*(X)$ is a nonempty compact connected subset of **C**, and $\Sigma^*(F) \subseteqq \Omega^*(F)$ $(F \varepsilon Q^*(X))$ [4]. Also we recall that a Banach space $(Y, \|\cdot\|)$ is said to be uniformly convex, if whenever $x_n \varepsilon Y$, $y_n \varepsilon Y$, $\|x_n\| \leqq 1$, $\|y_n\| \leqq 1$ and $\|x_n + y_n\| \to 2$, then $\|x_n - y_n\| \to 0$.

PROPOSITION 2.2. If X is uniformly convex and $F \varepsilon Q^*(X)$, then $\{\lambda \varepsilon \Omega^*(F) : |\lambda| = |F|^*\} \subseteqq \Sigma^*(F)$, where $|\cdot|^*$ denotes the seminorm on $Q^*(X)$[4].

PROOF. Let $\lambda \varepsilon \Omega^*(F)$and $|\lambda| = |F|^*$. We may assume that $\lambda \neq 0$, for otherwise $\tilde{F} = 0 \varepsilon \tilde{Q}^*(X)$, the normed space of all equivalence classes of B*-quasibounded maps, i.e.,

$$\tilde{Q}^*(X) = Q^*(X)/N(|\cdot|^*) \ [4]$$

and the result follows immediately. Since we may replace F by $\lambda^{-1}F$, there is no loss of generality in assuming that $|F|^* = \lambda = 1$.

Now, there exists $(x_n, f_n) \varepsilon \prod_n$ such that

$$\frac{f_n(<F(x_n, f_n), x_n>)}{\|x_n\|_X^2 \|f_n\|} \to 1$$

as $n \to \infty$ and therefore

$$\frac{f_n(<(\pi + F)(x_n, f_n), x_n>)}{\|x_n\|_X^2 \|f_n\|} \to 2. \qquad (1)$$

Since

$$1 + \frac{\|F(x_n, f_n)\|_X}{\|x_n\|_X} \geqq \frac{\|(\pi + F)(x_n, f_n)\|_X}{\|x_n\|_X}$$
$$\geqq \frac{|f_n(<(\pi + F)(x_n, f_n), x_n>)|}{\|x_n\|_X^2 \|f_n\|}, \qquad (2)$$

and $|F|^*=1$ it follows that

$$\left\| \frac{x_n}{\|x_n\|_X} + \frac{F(x_n, f_n)}{\|x_n\|_X} \right\|_X \to 2. \tag{3}$$

But (3) and X uniformly convex imply

$$\frac{\|(\pi - F)(x_n, f_n)\|_X}{\|x_n\|_X} \to 0. \tag{4}$$

Hence from (4) we obtain

$$d^*(F) = \lim_{r\to\infty} \inf_{\Pi_r} \frac{\|(\pi - F)(x, f)\|_X}{\|x\|_X} = 0,$$

i.e., $1 \varepsilon \Sigma^*(F)$.

On $W^*(X)$, the following seminorm is defined:

$$\omega^*(F) = \lim_{r\to\infty} \sup_{\Pi_r} \frac{|f(<F(x, f), x>)|}{\|x\|_X^2 \|f\|} \text{ [4]}.$$

PROPOSITION 2.3. The multivalued function $F \varepsilon W^*(X) \to \Omega^*(F)$ is upper semicontinuous, i.e., given an neighborhood V of $\Omega^*(F)$ there exists an $\varepsilon > 0$ such that $\Omega^*(G) \subset V$ for $G \varepsilon W^*(X)$, $\omega^*(F-G) < \varepsilon$.

PROOF. Suppose $\omega^*(G_n - F) \leqq \frac{1}{n}$, $z_n \varepsilon \Omega^*(G_n)$, $z_n \to z$. We will show that $z \varepsilon \Omega^*(F)$. It can be easily seen that this property implies the upper semicontinuity of $\Omega^*(F)$. By the definition of the seminorm $\omega^*(\cdot)$ we find $c_n > 0$ such that

$$|f(<G_n(x,f) - F(x,f), x>)| \leqq \left(\frac{2}{n}\right) \|x\|_X^2 \|f\|$$

for $(x, f) \varepsilon \Pi_0$, $\|x\|_X \geqq c_n$. By the definition of a B*-numerical range we find $(x_n, f_n) \varepsilon \Pi_0$, $\|x_n\|_X \geqq n + c_n$ such that

$$|f_n(<(z_n\pi - G_n)(x_n, f_n), x_n>)| \leqq \left(\frac{1}{n}\right) \|x_n\|_X^2 \|f_n\|.$$

Hence $|f_n(<(z\pi - F)(x_n, f_n), x_n>)| \leqq |f_n(<(F - G_n)(x_n, f_n),$

$x_n>)|$

$$+|f_n(<(G_n-z_n\pi)(x_n,\ f_n),\ x_n>)|+|z_n-z|\,\|x_n\|_X^2\|f_n\|$$
$$\leqq(\frac{3}{n}+|z_n-z|)\,\|x_n\|_X^2\|f_n\|.$$

Letting $n\to\infty$ we see that $z\varepsilon\Omega^*(F)$.

As a consequence, the set $\{F\varepsilon W^*(X):\Omega^*(F)\neq\phi\}$ is closed in $W^*(X)$. Also the multivalued function $F\varepsilon Q^*(X)\to\Sigma^*(F)$ is upper semi-continuous.

We recall that a continuous map $P:X_0=X-\{0\}\to X$ is said to be B-numerically bounded, if the map $F:\Pi_0\to X$ given by $F(x,f)=P(x)$ is B*-numerically bounded. In this case the numbers $\omega^*(F)$, $\alpha^*(F)$ and the B*-numerical range $\Omega^*(F)$ are denoted by $\omega(P)$, $\alpha(P)$ and $\Omega(P)$ respectively[4]. We denoted by $W(X)$ the vector space of all B-numerically bounded maps on X_0.

Let $S=\{x\varepsilon X:\|x\|_X=1\}$ be the unit sphere in X, and let $\Phi:S\to X$ be a continuous map on S, i.e., a vector field on S. We say that Φ is B-numerically bounded, if the map $\bar{\Phi}(x)=\|x\|_X\Phi(\|x\|_X^{-1}x)$, $x\neq 0$, is B-numerically bounded. In this case we let $\omega(\Phi)=\omega(\bar{\Phi})$, $\alpha(\Phi)=\alpha(\bar{\Phi})$ and $\Omega(\Phi)=\Omega(\bar{\Phi})$.

If we set $\Pi=\{(x,\ f)\ \varepsilon\ X\times B^*:\|x\|_X=\|f\|=f(<x,\ x>)=1\}$, then Π is a connected subset of $X\times B^*$ with the norm$\times$weak* topology[6].

Proposition 2.4. Let Φ be a B-numerically bounded vector field on S. Then

(a) $\omega(\Phi)=\sup\limits_{\Pi}|g(<\Phi(u),\ u>)|$.

(b) $\alpha(\Phi)=\inf\limits_{\Pi}|g(<\Phi(u),\ u>)|$.

(c) $\Omega(\Phi)=\{g(<\Phi(u),\ u>):(u,\ g)\ \varepsilon\,\Pi\}^-$.

PROOF. (a) and (b) follow from

$$\frac{f(<\bar{\phi}(x),\ x>)}{\|x\|_X^2\|f\|}=\frac{f(<\|x\|_X\Phi(\|x\|_X^{-1}x),\ x>)}{\|x\|_X^2\|f\|}$$
$$=g(<\Phi(u),\ u>),$$

where $u=\|x\|_X^{-1}x$, $g=\|f\|^{-1}f$ and $(u,\ g)\ \varepsilon\ \Pi$. Now (c) becomes evident.

PROPOSITION 2.5. Let F be a continuous mapping of S into X, and let $W_B(F)=\{f(<Fx,\ x>):(x,f)\ \varepsilon\ \Pi\}$. Then $W_B(F)$ is connected.

PROOF. This follows from Corollary 3.4[6].

As a conseqence we see that $\Omega(\Phi)$ coincides with the ~~closure $\overline{W_B(\Phi)}$ of the B-spatial numerical~~ range $W_B(\Phi)$ of a continuous map $\Phi:S\to X$.

3. A nonlinear version of Lumer's formula

In [6] Yang proved the Lumer's formula

$$\sup\ \mathrm{Re}\ W_B(T)=\lim_{\alpha\to 0+}\frac{\|I+\alpha T\|-1}{\alpha}$$

for any bounded linear operator T on X, where $W_B(T)$ denotes the B-spatial numerical range of T.

Our aim in this section is to prove a nonlinear verson of Lumer's formula for the class of Lipschitz maps. But before we do this, we are going to state an elementary result which is a generalization of the well known properties of the logarithmic norm for bounded linear operators on a Banach space.

LEMMA 3.1 [2]. Let Y be a Banach space, and let $C(Y)$ be a vector space of continuous maps $f:Y_0=Y-\{0\}\to Y$ such that $I\varepsilon C(Y)$. Let δ be a semi-norm defined on $C(Y)$

such that $\delta(I)=1$. If for every $f \varepsilon C(Y)$ we define

$$\delta'(f) = \lim_{\rho\to 0+} \frac{\delta(I+\rho f)-1}{\rho} \qquad (*)$$

then the limit (*) exists and satisfies the properties:

(a) $|\delta'(f)| \leqq \delta(f)$.

(b) $\delta'(\mu f)=\mu\delta'(f), \ \mu \geqq 0$.

(c) $\delta'(f+g) \leqq \delta'(f)+\delta'(g)$.

(d) $|\delta'(f)-\delta'(g)| \leqq \delta(f-g)$.

LEMMA 3.2. If $P \varepsilon W(X)$, then

$$\sup \operatorname{Re} \Omega(P) \leqq \omega'(P). \qquad (1)$$

PROOF. From the inequality

$$\operatorname{Re} \frac{f(<P(x),\ x>)}{\|x\|_X^2 \|f\|} \leqq \frac{1}{\rho} \left\{ \frac{|f(<x+\rho P(x),\ x>)|}{\|x\|_X^2\ f\|} - 1 \right\}, \ \rho > 0$$

and the obvious fact

$$\sup \operatorname{Re} \Omega(P) = \lim_{r\to\infty} \sup_{\Pi_r} \operatorname{Re} \frac{f(<P(x),\ x>)}{\|x\|_X^2 \|f\|},$$

we obtain

$$\sup \operatorname{Re} \Omega(P) \leqq \frac{\omega(I+\rho P)-1}{\rho}, \ \rho > 0. \qquad (2)$$

Now (1) follows, if in (2) we let $\rho \to 0^+$.

On the vector space $Q(X)$ of all quasibounded maps on X, the following seminorm is defined:

$$|P| = \lim_{\|x\|_X\to\infty} \sup \frac{\|T_x\|_X}{\|x\|_X} [3].$$

THEOREM 3.3. If $P: X \to X$ is a Lipschitz map, i.e., there exists $k > 0$ such that

$$\|P(x)-P(y)\|_X \leqq k\|x-y\|_X, \; x, y \varepsilon X, \tag{1}$$

then $\sup \operatorname{Re} \ \Omega(P)=\omega'(P)=|P|'$. (2)

PROOF. Since, clearly $\omega'(P) \leqq |P|'$, from the previous lemma we see that it suffices to show that

$$|P|' \leqq \sup \operatorname{Re} \ \Omega(P). \tag{3}$$

Let $\mu = \sup \operatorname{Re} \ \Omega(P)$ and $\mu_r = \sup \operatorname{Re} \ \overline{\phi_P(\Pi_r)}$ $(r>0)$, where ϕ_P is a continuous map given by

$$\Phi_P(x, f) = \frac{f(<P(x), x>)}{\|x\|_X^2 \|f\|}, (x, f) \ \varepsilon \Pi_0.$$

We have for $(x, f) \ \varepsilon \Pi_r \ (r>0)$

$$\begin{aligned} \frac{\|(I-\rho P)(x)\|_X}{\|x\|_X} &\geqq \left| \frac{f(<(I-\rho P)(x), x>)}{\|x\|_X^2 \|f\|} \right| \\ &= \left| 1-\rho \frac{f(<P(x), x>)}{\|x\|_X^2 \|f\|} \right| \\ &\geqq 1-\rho \ \operatorname{Re} \frac{f(<P(x), x>)}{\|x\|_X^2 \|f\|} \\ &\geqq 1-\rho \ \sup \operatorname{Re} \ \overline{\phi_P(\Pi_r)} \geqq 1-\rho\mu_r \end{aligned}$$

and using the fact $\lim_{r\to\infty} \mu_r = \mu$, we obtain

$$\frac{\|(I-\rho P)(x)\|_X}{\|x\|_X} \geqq 1-\rho\mu_r > 0, \quad \|x\|_X \geqq r, \tag{4}$$

for all $\rho>0$ sufficiently small.

If we apply (1) we obtain

$$\begin{aligned} \|x+\rho P(x)\|_X &\geqq \|x\|_X - \rho\|P(x)\|_X \\ &\geqq \|x\|_X - \rho(\|P(0)\|_X + k\|x\|_X) \\ &\geqq (1-k\rho)\|x\|_X - \rho\|P(0)\|_X. \end{aligned}$$

Thus, if we let $0<\rho<\frac{1}{k}$ we see from this last inequality that we can choose $\|x\|_X \geqq r$ large enough so that

$$\|x+\rho P(x)\|_X \geqq r.$$

Hence we can apply (4) with $x+\rho P(x)$ instead of x and obtain

$$\|(I-\rho P)(I+\rho P)(x)\|_X \geqq (1-\rho\mu_r)\|x+\rho P(x)\|_X,$$

and

$$\|(I+\rho P)(x)-\rho P(I+\rho P)(x)\|_X \geqq (1-\rho\mu_r)\|x+\rho P(x)\|_X. \quad (5)$$

From (1) we obtain

$$\begin{aligned}\|(I+\rho P)(x)-\rho P(I+\rho P)(x)\|_X &\leqq \|x\|_X+\rho\|P(x) \\ &\quad -P(I+\rho P)(x)\|_X \\ &\leqq \|x\|_X+\rho k\|x-(I+\rho P)(x\|_X) \\ &= \|x\|_X+\rho^2 k\|P(x)\|_X.\end{aligned}$$

Thus we have

$$\|(I+\rho P)(x)-\rho P(I+\rho P)(x)\|_X \leqq \|x\|_X+\rho^2 k\|P(x)\|_X. \quad (6)$$

From (5) and (6) we get

$$\|x\|_X+\rho^2 k\|P(x)\|_X \geqq (1-\rho\mu_r)\|x+\rho P(x)\|_X$$

and hence

$$1+\rho^2 k\frac{\|P(x)\|_X}{\|x\|_X} \geqq (1-\rho\mu_r)\frac{\|x+\rho P(x)\|_X}{\|x\|_X}. \quad (7)$$

If in (7) we take the lim sup as $r\to\infty$

we obtain $1+\rho^2 k|P| \geqq (1-\rho\mu)|I+\rho P|$,

and

$$\frac{|I+\rho P|-1}{\rho} \leqq \frac{\rho k|P|+\mu}{1-\rho\mu}. \quad (8)$$

If in (8) we let $\rho \to 0^+$, we obtain (3), and this completes the proof.

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