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Estimation of Pr(Y<X) in the Censored Case

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ABSTRACT

We study some estimation of the $\theta = Pr(Y<X)$ when the experiment is terminated before all of the items on the test have failed. We obtain maximum likelihood estimator and unique minimum variance unbiased estimator of θ. We consider asymptotic property of estimators and maximum likelihood estimator is compared with unique minimum veriance unbiased estimator in moderate sample size.

I. INTRODUCTION

Let the random variables X_1, X_2, ⋯, X_n and $Y_1, Y_2, \ldots, Y_m$ be independent and $F(x)$ be the common continuous cumulative distribution function (c.d.f) of X_i, $i = 1, 2, \cdots, n$ and $G(y)$ be the common c.d.f. of Y_i, $i = 1, 2, \cdots, m$. This paper concerns the estimation procedures for the probability function $\theta = P(Y \langle X)$.

This problem was proposed by Mann and Whitney [14]. They proposed a test statistic which could also be used for estimating unbiasedly the parameter θ. Birnbaum [3] showed that the Mann-Whitney statistic could be used in problems arising from practical situation and obtained a distribution free onesided confidence interval for θ. Sen [22] explored the possibility of arriving at some distribution free estimate of the variance of Mann-Whitney statistic based on the sample observations only and obtained the asymptotic confidence interval using this statistic. Also the problems have considered by Church and Harris [6], Muzumdar [15]. Govindarazuru [11], Downtown [7] and others.

In all previous studies, they have assumed that there was a complete sample available. But there are several situations where complete sample is neither possible nor desirable. In some cases the average failure time might be large giving a high probability of length of life test, especially the sample size is relatively large. These limit the number of items we can test. The practice of terminating a life test with only partial information available is called censoring of which there are two basic types *i.e* Type I (Time censored case), and Type II (Failure censored case). These problems were introduced by Epstein and Sobel [9], Mendenhall and harder [16], Boardman and Kendell [4], Boardman [5], Recently Yeum, J.K. and Kim, J.J. [24] proposed on estimator of θ in the Failure censored case. In this paper we

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deal with the estimation procedure of θ where the experiment is terminated before all of the items on the test have failed. In section 2, we are concerned with the maximum likelihood estimator (M.L.E.) of θ for the failure censored case and time censored case. Both cases are divided into two situations which are with replacement case and without replacement case. In section 3, we concern with the unique minimum variance unbiased estimator (UMVUE) of θ for the failure censored cases which are divided into two situations. In Section 4. We consider the asymptotic properties of estimators and the estimates of the mean square error (M.S.E.) and biases are obtained.

II. MAXIMUM LIKELIHOOD ESTIMATOR OF θ

2.1. Failure censored sample case

2.1.1. Without replacement case

Suppose n items are put to test and assume that the items which failed are not replaced by similar new ones. Let X and Y be independent random variables with exponential density functions.

$$f_1(x) = \alpha e^{-\alpha x} \text{ and } f_2(y) = \beta e^{-\beta y} \quad (1)$$

respectively.

Let $X_1 < X_2 < \ldots < X_r$ denote the failure times of r items and $(n-r)$ items be survived until x_r and also $Y_1 < \ldots < Y_s$ denote the failure times of s items and $(m-s)$ items be survived until y_s. Then the joint p.d.f of $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ are

$$f_1(x_1, x_2, \cdots, x_r : \alpha) = \frac{n!}{(n-r)!} \alpha^r \exp[-\alpha\{\sum_{i=1}^{r} x_i + (n-r)x_r\}]$$

and

$$f_2(y_1, y_2, \cdots, y_s : \beta) = \frac{m!}{(m-s)!} \beta^s \exp[-\beta\{\sum_{i=1}^{s} y_i + (m-s)y_s\}] \quad (2)$$

respectivety.

From (2), it is easily shown that the M.L.E. of α and β are

$$\hat{\alpha}_{r1} = \frac{r}{\sum_{i=1}^{r} x_i + (n-r)x_r} \quad \text{and}$$

$$\hat{\beta}_{s1} = \frac{s}{\sum_{i=1}^{s} y_i + (m-s)y_s} \quad (3)$$

So we obtain, by using the invariant property of M.L.E, the M.L.E of

$$\theta = P_r(Y < X) = \frac{\beta}{\alpha+\beta} \quad \text{is}$$

$$\hat{\theta}_1 = \frac{\hat{\beta}_{s1}}{\hat{\alpha}_{r1} + \hat{\beta}_{s1}} = (1 + \frac{\hat{\beta}^*_{s1}}{\hat{\alpha}^*_{r1}})^{-1} \quad (4)$$

where $\hat{\alpha}^*_{r1} = (\hat{\alpha}_{r1})^{-1} = \dfrac{\sum_{i=1}^{r} x_i + (n-r)x_r}{r}$

and $\hat{\beta}^*_{s1} = (\hat{\beta}_{s1})^{-1} = \dfrac{\sum_{i=1}^{s} y_i + (m-s)y_s}{s}$

We know that $\sum_{i=1}^{r} x_r + (n-r)x_r$ and $\sum_{i=1}^{s} y_s + (m-s)y_s$ are the total times of a test. It is easily known that $\hat{\alpha}^*_{r1}$ and $\hat{\beta}^*_{s1}$ are both sufficient statistics for α and β, respectively. In order to derive an optimal confidence interval for the θ, let $d_1 = r\hat{\alpha}^*_{r1} = \sum_{i=1}^{r} X_i + (n-r)x_r$ and $d_2 = s\hat{\beta}^*_{s1} = \sum_{i=1}^{s} y_i + (m-s)y_s$.

Then we know that $2\alpha d_1$ and $2\beta d_2$ are independent X^2 distribution with degree of freedom $2r$ and $2s$ respectively. Therefore, by the properties of X^2 distribution, $c = \dfrac{\beta d_2}{\alpha d_1 + \beta d_2}$ is distributed beta distribution with parameters s and r. Thus, for $c_1 < c < c_2$, the limit on c such that $P_r(c_1 < c < c_2) = I(c_2 : s, r) - I(c_1 : s, r) = 1 - \alpha$ may be converted to limit on θ where $I(c_2 : s, r) = \int_0^{c_2} B(s,r)\, dx$ and $B(s,r)$ denotes the p.d.f. of a beta distribution with parameters s and r. Hence the uniformly most accurate unbiased (U.M.A.U) confidence interval for θ with confidence coefficient $(1 - \alpha)$ is given by

$$\frac{c_1 d_1}{(1-c_1)d_2 + c_1 d_1} \le \theta$$

$$\le \frac{c_2 d_1}{(1-c_2)d_2 + c_2 d_1} \quad (5)$$

Now, letting $\rho_1 = 1 - c_2$, $\rho_2 = 1 - c_1$ and

$u = \frac{r\hat{\alpha}^*_{r1} - s\hat{\beta}^*_{s1}}{r\hat{\alpha}^*_{r1}}$, then

we immediately find that an alternative form of the confidence interval for θ is

$$\frac{1-\rho_2}{1-u\rho_2} \le \theta \le \frac{1-\rho_1}{1-u\rho_1} \quad (6)$$

2.1.2 With replacement case

In this subsection, we suppose that n items are initially placed on test and failed items are immediately replaced with new items. Let X and Y be independent random variable with p.d.f. (1) and suppose $X_{(1)}, \ldots, X_{(r)}$ and $Y_{(1)}, \ldots, Y_{(s)}$ are the r and s ordered failure times respectively. Then these failures may be considered as occurrences of a Poisson processes with parameter $n\alpha$ and $m\beta$. Thus, the interarrival times $W_i = X_{(i)} - X_{(i-1)}$ and $V_i = Y_{(i)} - Y_{(i-1)}$, are independent exponential variables with parameters $n\alpha$ and $m\beta$. Therefore, the likelihood function of w and v are

$$L_1(w_1, w_2, \cdots, w_r : \alpha) = (n\alpha)^r \exp[-n\alpha \sum_{i=1}^{r} w_i]$$

and

$$L_2(v_1, v_2, \cdots, v_s : \beta) = (m\beta)^s \exp[-m\beta \sum_{i=1}^{s} v_i] \quad (6)$$

respectively. Inverse transforms of W_i and V_i are $X_{(i)} = \sum_{j=1}^{i} W_j$ and $Y_{(i)} = \sum_{j=1}^{i} V_i$. We obtain following likelihood functions of $X_{(i)}$ and $Y_{(i)}$;

$$L_1(x_{(1)}, x_{(2)}, \cdots, x_{(r)} : \alpha) = (n\alpha)^r \exp[-n\alpha\ x_{(r)}],\ 0 < x_{(1)} < \cdots < x_{(r)} < \infty$$

and

$$L_2(y_{(1)}, y_{(2)}, \cdots, y_{(s)} : \beta) = (m\beta)^s \exp[-m\beta\ y_{(s)}],\ 0 < y_{(1)} < \cdots < y_{(s)} < \infty \quad (7)$$

Therefore, we obtain the M.L.E. of α and β;

$$\hat{\alpha}_{r2} = \frac{r}{nx_{(r)}} \quad \text{and} \quad \hat{\beta}_{s2} = \frac{s}{my_{(s)}}$$

Hence,

$$\theta_2 = \frac{\hat{\alpha}^*_{r2}}{\hat{\alpha}^*_{r2} + \hat{\beta}^*_{s2}} \quad (8)$$

where $\hat{\alpha}^*_{r2} = (\hat{\alpha}_{r2})^{-1} = \frac{nx_{(r)}}{r}$ and

$$\hat{\beta}^*_{s2} = (\hat{\beta}_{s2})^{-1} = \frac{my_{(s)}}{s}$$

Here, $nx_{(r)}$ and $my_{(s)}$ are also the total test time occurred in the experiment, respectively. Since $x_{(r)}$ and $y_{(s)}$ are equivalent to a sum of independent exponential variables and $x_{(r)}$ and $y_{(s)}$ are a complete sufficient statistic of α and β, we know that $2r\hat{\alpha}^*_{r2}$ and $2s\hat{\beta}^*_{s2}$ are independent central X^2-distribution with $2r$ and $2s$ degree of freedom, respectively. Thus, in the expriment of two types (with or without replacement), the M.L.Es of α are unbiased and have the same variance. Similarly, the M.L.Es of β have same result.

2.2. Time censored sample case

2.2.1 Without replacement case

In this section, we consider the case of time censored. The time censored samples arise when we terminate the life testing experiment at a preassigned time t_{01}. Here the number of items of X that failed before preassigned time t_{01} is a random variable which we denote R. Let $P_1(t_{01})$ be the probability of failure before t_{01} and $(n-r)$ items that survived beyond t_{01}, then the R has binomial distribution. Similarly, the number of items of Y, S that failed before time t_{02} has the binomial distributions.

$$P(R=r) = \binom{n}{r} p_1^r (1-p_1)^{n-r}$$

$$r = 0, 1, 2, \cdots, n$$

and $P(S=s)=\binom{m}{s} p_2^s(1-p_2)^{m-s}$,

$$s=0,1,2,\cdots,m \quad (9)$$

where $p_1=P_{01}(t_{01})=1-\exp(-\alpha t_{01})$ and $p_2=P_{02}(t_{02})=1-\exp(-\beta t_{02})$ respectively. We consider in this subsection without replacement case. The conditional p.d.fs of X and Y given that the items have failed before time t_{01} and t_{02} are given by

$$h_1(x\mid\alpha)=\begin{cases}\dfrac{\alpha\exp(-\alpha x)}{1-\exp(-\alpha t_{01})}, & 0<x<t_{01}\\ 0, & \text{otherwise}\end{cases}$$

and

$$h_2(y\mid\beta)=\begin{cases}\dfrac{\beta\exp(-\beta y)}{1-\exp(-\beta t_{02})}, & 0<y<t_{02}\\ 0, & \text{otherwise}.\end{cases} \quad (10)$$

That is, given $R=r$ and $S=s$, the observation is equivalent to a random sample from a truncated exponential distribution. Thus for $r>0$ and $s>0$, the joint p.d.fs of $X_1<X_2<\ldots<X_r$ and $Y_1<Y_2<\ldots<Y_s$ are given by

$$g_1(x_1,\cdots,x_r\mid r)=r!\,\alpha^r\frac{\exp(-\alpha\sum x_i)}{\{1-\exp(-\alpha t_{01})\}^r}$$

and

$$g_2(y_1,\cdots,y_s\mid s)=s!\,\beta^s\frac{\exp(-\beta\sum y_i)}{\{1-\exp(-\beta t_{02})\}^s} \quad (11)$$

respectively.

Thus the likelihood functions of samples are the joint p.d.fs of $X_1, X_2, \cdots, X_r$, R and $Y_1, \cdots, Y_s$, S. Hence,

$$L_1\{x_1,x_2,\cdots,x_r,r\mid\alpha\}=g_1(x_1,\cdots,x_r\mid r)\cdot P(R=r)=\frac{n!}{(n-r)!}\alpha^r\exp[-\alpha\{\sum_{i=1}^r x_i+(n-r)t_{01}\}].$$

and

$$L_2\{y_1,y_2,\cdots,y_s,s\mid\beta\}=g_2(y_1,\cdots,y_s\mid s)\cdot P(S=s)=\frac{m!}{(m-s)!}\beta^s\exp[-\beta\{\sum_{i=1}^s y_i+(m-s)t_{02}\}]. \quad (12)$$

From (12), we obtain the M.L.Es of α and β are

$$\hat{\alpha}_{r3}=\frac{r}{\sum_{i=1}^r x_i+(n-r)t_{01}} \quad \text{and}$$

$$\hat{\beta}_{s3}=\frac{s}{\sum_{i=1}^s y_i+(m-s)t_{02}} \quad (13)$$

respectively.

Hence, we obtain the M.L.E. of θ for the Type I case

$$\hat{\theta}_3=\frac{\hat{\beta}_{s3}}{\hat{\alpha}_{r3}+\hat{\alpha}_{s3}}=\frac{1}{\dfrac{r\{\sum y_i+(m-s)t_{01}\}}{s\{\sum x_i+(n-r)t_{02}\}}+1} \quad (14)$$

From (11), it is clear that $\hat{\alpha}_{r3}$ and r, $\hat{\beta}_{s3}$ and s are joint sufficient statistics for α, β. Also, $\sum_{i=1}^r x_i$ and r, $\sum_{i=1}^s y_i$ and s can be considered joint sufficient statistics, however, it is not completely clear how to develop optimum procedures in these cases since two sufficient statistics are required for a single parameter. Since R and S are random variables and the numerate of (13) are a sum of random number of random variables, the distribution theory of $\hat{\alpha}_{r3}$, $\hat{\beta}_{s3}$ in the time censored case is quite complicated. Thus we don't find the exact p.d.f. of $\hat{\theta}_3$ yet, the problems of obtaining the properties of $\hat{\theta}_3$ for the time censored case remains unsolved. Bartholomew [1] proposed the approximation method for these problems.

2.2.2 With replacement case

Suppose we carry out a time-censored test where the items that failed are immediately replaced and n positions are available in the test equipment, then for any given position the failure times corresponds to the occurrences of a Poisson process with parameter α. It follows from the properties of Poisson processes that if n independent Poisson processes with parameter α are occurring simultaneously, then the occurrences may be considered to come from a single Poisson processes with parameter $n\alpha$. Consequently, the number of failures r and s have a Poisson distribution with parameter

$\lambda_1 = n\alpha t_{01}$, $\lambda_2 = m\beta t_{02}$.

We write

$$P(r \mid \alpha) = \frac{e^{-n\alpha t_{01}}(n\alpha t_{01})^r}{r!}, \quad r = 0,1,2,\cdots$$

and

$$P(s \mid \beta) = \frac{e^{-m\beta t_{02}}(m\beta t_{02})^s}{s!}, \quad s = 0,1,2,\cdots \quad (15)$$

Given that r and s failures have occurred, the failure times are conditionally distributed as ordered uniform variables,

$$h_1(x_1, \cdots, x_r, r) = \frac{r!}{(t_{01})^r}, \quad 0 < x_1 < \cdots < x_r < t_{01}, \quad r = 1,2,\cdots$$

and

$$h_2(y_1, \cdots, y_s, s) = \frac{s!}{(t_{02})}, \quad 0 < y_1 < \cdots < y_s < t_{02}, \quad s = 1,2,\cdots \quad (16)$$

respectively.

The likelihood functions in this case are

$$L_1(x_1, \cdots, x_r : r) = h_1(x_1, \cdots, x_r \mid r) P(R = r) = (n\alpha)^r \exp(-n\alpha t_{01}), \quad 0 < x_1 < \cdots < x_r < t_{01}, \quad r = 1,2,\cdots$$

and

$$L_2(y_1, \cdots, y_s : s) = h_2(y_1, \cdots, y_s \mid s) P(S = s) = (m\beta)^s \exp(-m\beta t_{02}), \quad 0 < y_1 < \cdots < y_s < t_{02}, \quad s = 1,2,\cdots \quad (17)$$

Thus, the M.L.E of α and β are easily found to be

$$\hat{\alpha}_{r4} = \frac{r}{nt_{01}} \quad and \quad \hat{\beta}_{s4} = \frac{s}{mt_{02}}, \quad r > 0, s > 0.$$

Therefore, the M.L.E of θ,

$$\hat{\theta}_4 = \frac{\hat{\beta}_{s4}}{\hat{\alpha}_{r4} + \hat{\beta}_{s4}} = \frac{1}{\frac{nt_{02}}{mt_{01}} \cdot \frac{r}{s} + 1} \quad (18)$$

III. MINIMUM VARIANCE UNBIASED ESTIMATOR (M.V.U.E) OF θ

3.1. Without replacement case

Since the distribution of $\hat{\theta}_3$ did not find in the time censored case, we consider only the case of failure censored case. From the point of view of the sampling theory, it is clear that $\hat{\alpha}^*_{r1}$ and $\hat{\beta}^*_{s1}$ are sufficient statistic for α and β, respectively. Also, we know that these are a complete sufficient statistic.

Let $z_{1i} = (n - i + 1)(x_i - x_{i-1})$. Then $(z_{11}, \cdots, z_{1r})$ is $i.i.d$ and its distribution is

$$f(z_1) = \alpha e^{-\alpha z_1}, \quad z_1, \alpha > 0. \quad (19)$$

Similarly, let $z_{2i} = (m - i + 1)(y_i - y_{i-1})$, then z_{2i} has the distribution as following;

$$f(z_2) = \beta e^{-\beta z_2}, \quad z_2, \beta > 0, \quad (20)$$

Hence $r\alpha\hat{\alpha}^*_{r1}$ and $s\beta\hat{\beta}^*_{s1}$ are both the gamma distribution. Using the Blackwell-Rao and Lehmann-Scheffẽ theorem, the unique M.V.U.E. of

θ is obtained by taking the conditional expectation of $I(Y, X)$ given $(\hat{\alpha}^*_{r_1}, \hat{\beta}^*_{s_1})$

$$\tilde{\theta}_1 = E[\, I(Y_1, X_1) \mid \hat{\alpha}^*_{r1}, \hat{\beta}^*_{s1} \,]$$

$$= \iint_A I(y_1, x_1)\, h(x_1, y_1 \mid \hat{\alpha}^*_{r1}, \hat{\beta}^*_{s1})\, dx_1\, dy_1$$

$$= \iint_A I(y_1, x_1)\, f(x_2 \mid \hat{\alpha}^*_{r1})\, g(y_1 \mid \hat{\beta}^*_{s1})\, dx_1\, dy_1 \qquad (21)$$

where A is the sample space in which both $f(x_1 \mid \hat{\alpha}^*_{r1})$ and $g(y_1 \mid \hat{\beta}^*_{s1})$ are nonzero and

$$I(Y, X) = \begin{cases} 0 & if \quad Y > X \\ 1 & if \quad Y < X \end{cases}$$

i) For the case $s\hat{\beta}^*_{s1} \leqq r\hat{\alpha}^*_{r1}$,

$$\tilde{\theta}_1 = {}_2F_1\left(1-r, 1 : s : \frac{s\hat{\beta}^*_{s1}}{r\hat{\alpha}^*_{r1}}\right)$$

$$= \sum_{i=0}^{r-1} (-1)^i \frac{(s-1)!(r-1)!}{(s+i-1)!(r-i-1)!} \left(\frac{s\hat{\beta}^*_{s1}}{r\hat{\alpha}^*_{r1}}\right)^i \qquad (22)$$

where ${}_2F_1(a, b : c : x)$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-cx)^{-a}\, dt$$

$$= \sum_{j=1}^{\infty} \frac{a_{(j)}\, b_{(j)}}{c_{(j)}} \cdot \frac{x^j}{j!}$$

$$,\ c > b > 0,\ a > 0,\ x < 1.$$

and $d(k) = d(d+1)\cdots(d+k-1)$.

ii) For the case $s\hat{\beta}^*_{s1} > r\hat{\alpha}^*_{r1}$,

$$\tilde{\theta}_1 = 1 - {}_2F_1\left(1-s : 1 : r : \frac{r\hat{\alpha}^*_{r1}}{s\hat{\beta}^*_{s1}}\right)$$

$$= 1 - \sum_{i=0}^{s-1} (-1)^i \frac{(s-1)!(r-1)!}{(s-i-1)!(r+i-1)!} \cdot \frac{r\hat{\alpha}^*_{r1}}{s\hat{\beta}^*_{s1}} \qquad (23)$$

3.2. With replacement case

Since $\hat{\alpha}^*_{r2}$, $\hat{\beta}^*_{s2}$ are both a complete sufficient statistic for α and β respectively, we can use the Blackwell-Rao and Lehmann-scheffẽ theorem. Let $W_i = X_{(i)} - X_{(i-1)}$, then $\sum_{i=1}^{r} W_i = X_{(r)}$. Therefore $r\alpha\hat{\alpha}^*_{r2}$ has the gamma distribution $\Gamma(r, 1)$. Similarly, $\sum_{i=1}^{s} V_i = Y(s)$ has the gamma distribution $\Gamma(s, 1)$. From the fact $\hat{\alpha}^*_{r2} = \frac{nx_{(r)}}{r} = \frac{n\sum w_i}{r}$

$= n\bar{w}$, and $\hat{\beta}^*_{s2} = m\bar{v}$, let $\bar{w}^* = \frac{n\sum_{i=2}^{r} w_i}{r-1}$

and $\bar{v}^* = \frac{m\sum_{i=2}^{s} v_i}{s-1}$. Then $\alpha(r-1)\bar{w}^*$ has $\Gamma(r-1, 1)$ and $\beta(s-1)\bar{v}^*$ has $\Gamma(s-1, 1)$ and w_1 and $\bar{w}^*$ are independent and v_1 and $\bar{v}^*$ are independent. Thus the joint p.d.f.

$$f(w_1, \bar{w}^*) = \frac{\alpha^r (r-1)^{r-1} \cdot n}{\Gamma(r-1)} (\bar{w}^*)^{r-2} \exp[-\alpha\{(r-1)\bar{w}^* + nw_1\}] \qquad (24)$$

since $rn\bar{w} = (r-1)\bar{w}^* + nw_1$,

$$\bar{w}^* = \frac{nr\bar{w} - nw_1}{r-1},$$

and $$f(w_1, n\bar{w}) = \frac{\alpha^r \cdot n}{\Gamma(r-1)} (nr\bar{w} - nw_1)^{r-2} \exp\{-\alpha(nr\bar{w})\}. \qquad (25)$$

Therefore, the conditional p.d.f. given $n\bar{w}$,

$$f(w_1 \mid n\bar{w}) = \frac{f(w_1, \hat{\alpha}^*_{r2})}{f(\hat{\alpha}^*_{r2})}$$

$$= \frac{(r-1)}{x_{(r)}}\left[1 - \frac{w_1}{x_{(r)}}\right]^{r-2},$$

$$0 < w_1 < x_{(r)} = \frac{r}{n} \hat{\alpha}^*_{r2}. \qquad (26)$$

similarly,

$$g(u_1 \mid m\bar{v}) = \frac{s-1}{y_{(s)}}\left(1 - \frac{u_1}{y_{(s)}}\right)^{s-2},$$

$$0 < u_1 < y_{(s)} = \frac{s}{m} \hat{\beta}^*_{s2}$$

By taking the conditional expectation of $I(Y,X)$ given $(\hat{\alpha}^*_{r2}, \hat{\beta}^*_{s2})$, we can obtain the UMVUE of θ.

i) For the case $y_{(s)} \leq x_{(r)}$

$$\tilde{\theta}_2 = {}_2F_1(1-r, 1 : s : z), \quad z = \frac{y_{(s)}}{x_{(r)}}$$

$$= \sum_{i=0}^{r-1} (-1)^i \frac{(s-1)!(r-1)!}{(s+i-1)!(r-i-1)!} \left(\frac{y_{(s)}}{x_{(r)}}\right)^i \qquad (27)$$

ii) For the case $y_{(s)} > x_{(r)}$

$$\tilde{\theta}_2 = 1 - {}_2F_1\left(1-s, 1 : r : \frac{x_{(r)}}{y_{(s)}}\right) \qquad (28)$$

IV. ASYMPTOTIC DISTRIBUTION AND EMPIRICAL COMPARISON.

In [24], we investigated the large sample distribution of the $\hat{\theta}$ in without replacement censored case and we showed that M.V.U.E $\tilde{\theta}$ converge almost surely to M.L.E. $\hat{\theta}$ and the limiting distributions are also held for $\tilde{\theta}$. For the with replacement case, the distribution of estimator are same as the distribution of estimator for the without replacement case. Therefore same asymptotic distributions are obtained (see [24] pp. 266).

Since the exact distribution of the estimates is very difficult to handle, we investigate their relative performance in a moderate sample n=20, m=20 through Monte carlo simulation. In this investigation, we assume that the experiments are terminated after the first 15 items failed, respectively. Estimates of the mean square error (M.S.E.) and bias were obtained from 200 trials for the two items with $\lambda = \frac{\beta}{\alpha} = 1, 2, 3, 4, 5$. In each situation, a trial consisted of generating 40 uniform (0.1) random numbers U_i and transformed to $X_i = -\lambda \log U_i$, $i \leq 20$, and $Y_i = -\log U_i$, $i = 21, . . , 40$. Using the first 15 items, the value of the MLE $\hat{\theta}$ was obtained and the value of UMVUE $\tilde{\theta}$ was obtained and the total value of θ was computed for each λ. The results on the estimated MSE and Bias appear in table. Although $\tilde{\theta}$ is known to be unbiased, its estimated bias is recorded as a check on the computations. Also the MLE is biased, it is apparant that in all cases included in the study, the magnitude of $(\text{bias})^2$ is negligible relative to the M.S.E. Furthermore, the M.S.E of both estimates appear to be nearly equal.

λ	θ	Bias		MSE	
		MLE	MVUE	MLE	MUVE
1	0.500	0.006	0.006	0.009	0.009
2	0.666	0.007	0.003	0.008	0.007
3	0.750	0.002	0.003	0.005	0.004
4	0.800	0.002	0.004	0.003	0.003
5	0.833	0.004	0.005	0.002	0.002

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