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Generalized Control Procedure for a Process Subject to Linear Shift

Hang Rae Kim* Joong Soon Jang*

ABSTRACT

A model for obtaining optimal resetting period and optimal target value of the process quality level is proposed when the quality characteristic is subject to a linear shift in mean. Different sales conditions are considered where good items can be sold at a regular price and bad items at a discounted one. Some numerical examples are also given.

I. Introduction

This paper deals with the problem of obtaining an optimal resetting period and selecting an optimal starting quality level when the quality characteristic of the product is subject to a systematic shift. Such shifts may be found in tool wear in machining, drawing, stamping and moulding operations, which make the process quality level to deteriorate over time. Thus, whenever the measured output characteristic reaches a certain undesired value, the process needs to be reset and a correction action should be performed. Considerable savings can be achieved in many industrial processes if optimal decision rules can be found for the starting value of the process and for when to shut it down for resetting the control parameters.

Gibra[4] obtains the optimal production run by controlling the initial setting for stable and unstable processes. He uses a control chart approach, assumes a constant variance and a linear shift in the mean, and provides solutions for both the single and the two-sided specification cases. The optimal control involves minimizing resetting cost and loss due to defective items. Kamat[6] provides a Bayes control procedure when there is a linear shift in the mean to detect the point at which the process goes outside of the desired specification limits and correction action must be initiated. Arcelus et al.[1] generalized Gibra's results to a case where both the mean and variance of the quality characteristic are subject to a linear or nonlinear shift.

In this paper, we extend Gibra's results to a case where the defective items can be either be

* Department of Industrial Engineering, Ajou University

sold at a reduced unit price on a secondary market or considered as scrap and reprocessed. (See, for example, Carlsson[3] or Bisgaard et al.[2].) The net profit is, then, a function of the income from products sold at a regular price, the income from products sold on the secondary market, and the production costs. Hence, the optimal control parameters should be obtained through maximizing the net profit function to the producer.

We assume here that the quality characteristic is normally distributed with a linearly decreasing mean over time and a constant variance and has one lower specification limit. Furthermore, we assume that the income function is a piecewise linear function of the quality characteristic and all items are inspected.

II. Notations and Assumptions

Notations

- X r.v denoting the quality characteristic
- μ_0 starting process mean (target value)
- θ the drift in the process mean per unit time $\theta \leq 0$
- μ_t process mean at time t $\mu_t = \mu_0 + \theta_t$
- σ^2 variance of the quality characteristic
- L lower specification limit
- b_a regular price for good items
- b_r reduced price for bad items ($< b_a$)
- c_r price reduction factor proportinal to the deficit of quality
- b_0 fixed production cost
- c_0 variable production cost
- τ length of resetting period
- K resetting cost

Assumptions

1. The quality characteristic X is normally distributed with mean μ_t and variance σ^2 at time t.
2. One item is produced per unit time.
3. Good items can be sold at a price b_a and bad items at a price $b_r + c_r\ (X-L)$.
4. The production cost for an item is $b_0 + c_0\ (X-L)$.
5. The resetting is made at a cost of K for every production interval of length.
6. The planning horizon is infinite.
7. The time to reset the process is negligible.

III. Model

Under the assumptions made above, we should seek for the optimal values of τ and μ_0 so that a longer production run is achieved and the rate of defective items is reduced. In many situations, however, the resetting period is usually specified, say, a day, a week, or a month. Moreover, in some cases, physical limitations of machines prevent the quality level from being higher than a prespecified one. Thus, three cases are considered in this paper: a model to obtain an optimal target value of μ_0 when the resetting period is held fixed, a model to obtain the resetting period with a prespecified target value and a model to obtain the resetting period and the target value simultaneously.

III-1. Fixed Resetting Period

The producer's income function per item with quality characteristic X is

$$\begin{cases} b_a & , X \geq L , \\ b_r + c_r(X-L) , & X < L . \end{cases}$$

Since the cost of producing an item with quality characteristic X is $b_0 + c_0(X-L)$, the net profit function for an item is

$$N(X) = \begin{cases} b_a - b_0 - c_0(X-L) & , X \geq L , \\ b_r - b_0 - (c_0 - c_r)(X-L) , & X < L . \end{cases}$$

This function can be simplified by writing

$$N(X) = \begin{cases} a - c_0(X-L) , & X \geq L , \\ r - c_0 p(X-L) , & X < L , \end{cases}$$

where $a = b_a - b_0$, $r = b_r - b_0$ ($< a$), and $1\text{-}p = c_r/c_0$. The parameter $1\text{-}p$ is the customer's compensation, besides the price reduction $a\text{-}r$, per unit measure of the quality characteristic. There are no formal restriction on p. However, p is assumed to be nonnegative and not greater than 1 from practical considerations.

The expected profit to the producer for one resetting cycle is then given by

$$C(\mu_0) = \int_0^\tau \mathrm{EN}(X)\,dt - K$$
$$= \frac{1}{\sigma}\int_0^\tau \int_{-\infty}^L \{ r - c_0 p (X - L) \} \phi((x - \mu_0 - \theta t)/\sigma)\,dx\,dt$$
$$+ \frac{1}{\sigma}\int_0^\tau \int_L^\infty \{ a - c_0 (X - L) \} \phi((x - \mu_0 - \theta t)/\sigma)\,dx\,dt\,, \quad \cdots\cdots (1)$$

where $\phi(\cdot)$ is the standard normal density function. Let $y_t = (L - \mu_0 - \theta t)/\sigma$. Then, (1) reduces to

$$C(\mu_0) = \int_0^\tau \{ (a + c_0 \sigma y_t) - (a - r)\Phi(y_t) - c_r (y_t \Phi(y_t) + \phi(y_t)) \} dt - K, \quad (2)$$

where $\Phi(\cdot)$ is the standard normal distribution function. Now, differentiating (2) with respect to μ_0, we have

$$C'(\mu_0) = \int_0^\tau \{ - c_0 + \frac{a - r}{\sigma} \phi(y_t) + c_r \Phi(y_t) \} dt. \quad \cdots\cdots\cdots\cdots (3)$$

Therefore, the optimal value of μ_0 can be obtained from the following equation

$$(a - r)[\Phi(y_0) - \Phi(y_\tau)] + c_r \sigma [y_0 \Phi(y_0) - y_\tau \Phi(y_\tau) + \phi(y_0) - \phi(y_\tau)] = c_0 \theta \tau. \quad \cdots\cdot (4)$$

We could not prove the uniqueness of the solution satisfying (4) in general. In practical problems, however, it is undesirable to keep the process mean outside the specification limit, *i.e.* $y_\tau > 0$. Hence, we can assume without loss of generality that

$$y_t \leq 0 \quad \text{for} \quad 0 \leq t \leq \tau \quad \cdots\cdots\cdots (5)$$

and

$$\phi(y_0) \leq \phi(y_\tau).$$

Under (5), we can easily show that $C''(\mu_0)$ is a nonnegative function of μ_0. Therefore, if the cost parameters satisfy

$$\Phi(\theta\tau/\sigma)[(a - r)/\theta + c_r \tau] - c_r \sigma [\phi(o) - \phi(\theta\tau/\sigma)]/\theta > c_0 \tau + (a - r)/2\theta,$$

there always exists an unique solution to (4). Otherwise, the optimal target value of μ_0 should be $\mu_0{*} = L - \theta\tau$.

Example 1.

Steel beams with lower specification limit for width $L = 12.3$ unit length are produced at a cost of $c_0 = 4.50$. The selling price is $a = 52.20$ and the scrap iron price is $r = 15.90$ and the cost reduction factor is $c_r = 0.7\,c_0$. The standard deviation of the beam length is $\sigma = 0.403$. The drift of the mean is $\theta = -0.02\sigma$ per unit time and the resetting cost is $K = 30000$. If the length of the resetting period is 800 unit time, we can obtain the optimal starting process mean $\mu_0{*} = 16.41457$ unit length.

III-2. Fixed Target Value

Suppose now that the initial value of the mean of the quality characteristic is μ_0. In this case, we should maximize the expected profit per unit time rather than maximizing the expected profit over resetting period.

From (2), the expected profit per unit time can be derived as follows.

$$C(\tau) = \frac{1}{\tau}[\int_0^\tau \{ a + c_0 \sigma y_t - (a - r)\Phi(y_t) - c_r [y_t \Phi(y_t) + \phi(y_t)] \} dt - K]. \quad (6)$$

Differentiating (6) with respect to τ gives

$$\partial C(\tau)/\partial \tau = \frac{1}{\tau^2}[(r - a)\int_0^\tau t\,\phi(y_t)\,dt + c_r \int_0^\tau t\,\Phi(y_t)\,dt - K] - c_0\theta/\tau. (7)$$

Hence, the optimal value of τ can be obtained from the following equation.

$$(\Phi(y_\tau)-\Phi(y_0))[2(a-r)y_0+c_r y_0^2+c_r]+\phi(y_\tau)(c_r(y_\tau-2y_0)+2(a-r))+\phi(y_0)(c_r y_0-2(a-r))=\frac{\theta^2}{\sigma^2}(2K+c_0\theta\tau^2-c_r\Phi(y_\tau)). \quad \cdots\cdots\cdots(8)$$

It is difficult to show that the value of τ satisfying (8) is the unique optimal solution. In real situations, however, it seems obvious that the optimal solution is unique and satisfies (8): As τ increases, resetting cost decreases and production cost increases. As τ decreases, resetting cost increases and production cost decreases.

Example 2.

Suppose now that the target value of μ_0 is 16.41457. Solving equation (8) by the well known 'bisection method' with initial values 0 and $(L-\mu_0)/\theta'$ we obtain the optimal length of the resetting period $\tau*=338.58420$.

III-3. Unspecified Resetting Period and Target Value

Now consider the case where both the resetting period and the target value are not specified. In this case, we should obtain their values simultaneously.

The expected profit per unit time can be derived easily from (6).

$$C(\mu_0,\tau)=\frac{1}{\tau}[\int_0^\tau\{a+c_0\sigma y_t-(a-r)\Phi(y_t)-c_r[y_t\Phi(y_t)+\phi(y_t)]\}dt-K]. \quad \cdots\cdots\cdots(9)$$

Taking the derivatives, we have

$$\frac{\partial C(\mu_0,\tau)}{\partial\mu_0}=\frac{1}{\tau}[\int_0^\tau\{-c_0+\frac{a-r}{\sigma}\phi(y_t)+c_r\Phi(y_t)\}dt, \quad \cdots\cdots\cdots(10)$$

$$\frac{\partial C(\mu_0,\tau)}{\partial\tau}=\frac{1}{\tau^2}[(r-a)\int_0^\tau t\phi(y_t)dt+c_r\int_0^\tau t\Phi(y_t)dt-K]-c_0\theta/2. \quad \cdots\cdots\cdots(11)$$

Hence, the optimal values can be obtained from the following two equations.

$$(a-r)(\Phi(y_0)-\Phi(y_\tau))-c_r\sigma[y_0\Phi(y_0)-y_\tau\Phi(y_\tau)+\phi(y_0)-\phi(y_\tau)]=c_0\theta\tau, \quad (12)$$

$$(\Phi(y_\tau)-\Phi(y_0))(2(a-r)y_0+c_r y_0^2+c_r)+\phi(y_\tau)(c_r(y_\tau-2y_0)+2(a-r))+\phi(y_0)(c_r y_0-2(a-r))=\theta^2(2K+c_0\theta\tau^2-c_r\Phi(y_\tau))/\sigma^2. \quad \cdots\cdots\cdots(13)$$

Example 3.

Suppose now that neither the target value nor the resetting period is prespecified in Example 1. Equations (12) and (13) are to be solved simultaneously; solve (12) first with a suitable value of τ to obtain a solution μ_0 and solve (13) with μ_0 to obtain the value of τ. Then, obtain a solution μ_0 from (12) and repeat until the solutions seem to converge.

The following table showes the solutions μ_0 and τ, of (12) and (13).

From the Table, the optimal value of the resetting period is $\tau^*=3.48208$ and the optimal target value is $\mu_0^*=13.20251$.

Iteration	μ_0	τ
1	15.27442	338.58420
2	14.40879	212.69501
3	13.75675	111.16035
4	13.34090	34.71577
5	13.21400	6.30586
6	13.20317	3.64755
7	13.20258	3.49149
8	13.20251	3.48237
9	13.20251	3.48210
10	13.20251	3.48208
11	13.20251	3.48208

Table. Solutions of resetting period and target value

IV. Concluding Remarks

A model for obtaining optimal resetting period and optimal quality level for a deteriorating process is proposed with generalized cost parameters. We assumed that the drift of quality characteristic is known. If not known, it should be estimated from the produced items. Some time series approach may be adopted in monitoring the process quality level and predicting the point at which resetting the process is initiated.

References

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Appendix

The following formulas were used in calculating the profit function.

1. $\phi'(x) = -x\,\phi(x)$

2. $$\int_0^\tau \Phi(y_t)\,dt = \int_0^\tau \int_{-\infty}^{y_t} \phi(x)\,dx\,dt$$
$$= \int_{-\infty}^{y_\tau} \int_0^\tau \phi(x)\,dt\,dx + \int_{y_\tau}^{y_0} \int_0^{\frac{L-\mu_0-\sigma x}{\theta}} \phi(x)\,dt\,dx$$
$$= \frac{\sigma}{\theta}\left[y_0\Phi(y_0) - y_\tau\Phi(y_\tau) + \phi(y_0) - \phi(y_\tau) \right]$$

3. $$\int_0^\tau y_t\Phi(y_t)\,dt = \tau\left[y_\tau + \theta\tau/2\sigma \right]\Phi(y_\tau) + \frac{\sigma}{2\theta}(y_0{}^2 - 1)(\Phi(y_0) - \Phi(y_\tau)) + \frac{\sigma}{2\theta}(y_0\,\phi(y_0) - y_\tau\,\phi(y_\tau)).$$

4. $$\int_0^\tau \phi(y_t)\,dt = \frac{\sigma}{\theta}(\Phi(y_0) - \Phi(y_\tau))$$

5. $$\int_0^\tau t\,\phi(y_t)\,dt = -\left(\frac{\sigma}{\theta}\right)^2\left[y_0(\Phi(y_\tau) - \Phi(y_0)) + \phi(y_\tau) - \phi(y_0) \right].$$

6. $$\int_0^\tau t\,\Phi(y_t)\,dt = \frac{\tau^2}{2}\Phi(y_\tau) + \frac{\sigma^2}{2\theta^2}\left[(y_0^2 + 1)(\Phi(y_0) - \Phi(y_2)) + y_0\,\phi(y_0) + \phi(y_2)(y_\tau - 2y_0) \right].$$