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A Note on Periodic Replacement with Minimal Repair at Failure

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ABSTRACT

Periodic replacement model with minimal repair at failure is extended to the case where quantity purchases are possible. A recursive relationship among replacement intervals is obtained, which shows that replacement intervals are an increasing sequence due to the inventory carrying cost. Using the relationship, a procedure is given for determining how many units to purchase on each order and when to replace each unit after it has begun operating so as to minimize the total cost per unit time over an infinite time span. The problem can be simplified if equal replacement intervals are assumed, and the solution is very close to the solution of the unconstrained problem.

1. INTRODUCTION

Suppose that an operating equipment fails according to some probability law. When the equipment fails, it must be repaired or replaced to continue its operation. If an equipment having an IFR is not restored to its original state by repair, then the repair cost increases with its operating life due to the increase of repair frequency. Therefore, a sequence of equipments operating one after another is believed to be more economical than operating one equipment without replacing it. Thus to minimize the total cost per unit time, one wishes to determine how many equipments to purchase on each order and when to replace each equipment after it has begun operating.

The idea of minimal repair was introduced by Barlow and Hunter [1]. This idea is that if an equipment fails, a repair can be made which does restore it not to its original state (good as new) but to its operating condition immediately before failure (bad as old). The policy discussed by Barlow and Hunter is to perform minimal repairs up to a scheduled time T and the time T is the only decision variable. They obtained an expression for the expected cost per unit time and an integral equation for the optimal age T. However, in their policy they assumed implicitly one for one ordering and fixed ordering cost was included in every

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replacement.

This article examines the extension of the Barlow and Hunter model [1] to the case where quantity purchases are possible. The decision variables are order quantity per order and replacement intervals for the equipments.

2. SYMBOLS

$h(t)$ = failure rate at time t

$H(t)$ = $\int_0^t h(x)\,dx$ = cumulative hazard

c_f = expected cost of a minmal repair

c_p = expected cost of a replacement; this includes purchasing price of replacing item

c_h = inventory carrying cost per item per unit time

K = fixed ordering cost

Q = order quantity per order

T_{Qi} = replacement interval of i-th used item for a given order quantity Q $(i=1,2,\cdots\cdots,Q)$

$C(Q,\mathbf{T}_Q)$ = expected cost per unit time if order quantity is Q and replacement intervals are $\mathbf{T}_Q=(T_{Q1},T_{Q2},\cdots\cdots,T_{QQ})$

$C^*(Q)$ = the optimum cost per unit time for a given order quantity Q

Other symbols are defined as needed.

3. THE MODEL

In this policy, Q units are purchased per order, i-th used unit is replaced after using for time interval $T_{Qi}\,(i=1,2,\cdots,Q)$ with minimal repair for any intervening failures. The entire cycle repeats after $\sum_{i=1}^{Q} T_{Qi}$. The problem is to select order quantity Q and replacement intervals $\mathbf{T}_Q=(T_{Q1},T_{Q2},\cdots\cdots,T_{QQ})$ so as to minimize the total cost per unit time.

$$C(Q,\mathbf{T}_Q)=\{K+c_pQ+c_f\sum_{i=1}^{Q}H(T_{Qi})+c_h\sum_{i=1}^{Q}(Q-i)\,T_{Qi}\}/\sum_{i=1}^{Q}T_{Qi} \qquad (1)$$

The last two terms are expected repair and inventory carrying cost, since the expected number of failures in $[0,T]$ of a equipment under minimal repair is equal to cumulative hazard $H(T)$ [2, Chap. 4] and inventory level when i-th unit is operating is $(Q-i)$.

3.1. Determination of Replacement Intervals

To find the optimal replacement intervals for a given Q, we set the partial derivatives of Equation (1) with respect to T equal to zero, obtaining

$$c_fh(T_{Qi})+c_h(Q-i)=C(Q,T_Q) \qquad \text{for } i=1,2,\cdots,Q \qquad (2)$$

From Equation (2), we obtain the following recursive relationship among replacement intervals:

$$h(T_{Qi})=h(T_{Q,i+1})-c_h/c_f \qquad \text{for } i=1,2,\cdots,Q-1 \qquad (3)$$

Notice that a sequence of replacement intervals $\{T_{Qi}\}$ is an increasing sequence due to the inventory carrying cost c_h

Again summing Equation (2) over all i, we obtain

$$\sum_{i=1}^{Q}T_{Qi}\;c_fh(T_{Qi})+c_h(Q-i)$$
$$=(\sum_{i=1}^{Q}T_{Qi})\,C(Q,T_Q)$$
$$=K+c_pQ+c_f\sum_{i=1}^{Q}H(T_{Qi})+c_h\sum_{i=1}^{Q}(Q-i)T_{Qi}$$

or

$$\sum_{i=1}^{Q}\{h(T_{Qi})T_{Qi}-H(T_{Qi})\}=(K+c_pQ)/c \qquad (4)$$

Using Equation (3), we can express T in terms of T :

$$h(T_{Qi})=h(T_{QQ})-c_h(Q-i)/c_f$$

or

$$T_{Qi}=h^{-1}\{h(T_{QQ})-c_h(Q-i)/c_f\} \qquad (5)$$

Substituting Equation (5) into Equation (4), we can determine T^*_{QQ}. Once T^*_{QQ} is obtained, we can determine T^*_{Qi} using Equation (5) and from Equation (2) the optimum cost is

$$C^*(Q) = c_f h\ (T^*_{QQ}) \tag{6}$$

Thus the optimal solution satisfies Equations (4) and (5), and the optimum cost is given by Equation (6). If $Q=1$, Equations (4), (5) and (6) degenerate to Barlow and Hunter's.

REMARK: If $h(t)$ is strictly increasing, then there exists a unique replacement intervals T^*_{Q1}, T^*_{Q2}, ……, T^*_{QQ} (possibly infinite, that is, never to replace) satisfying Equations (4), (5), (6) and it must yield $C^*(Q)$ since the hessian matrix evaluated at the critical point is diagonal with elements $c_f h'(T^*_{Qi}) / \sum_{i=1}^{Q} T^*_{Qi}$ and thus the determinant is $(c_f / \sum_{i=1}^{Q} T^*_{Qi})^Q \prod_{i=1}^{Q} h'(T^*_{Qi}) > 0$.

3.2. Determination of Order Quantity

In order to find the condition that stocking policy is required, let us examine when $C^*(1) \geq C^*(2)$. in case of IFR, from Equation (6), $C^*(1) \geq C^*(2)$ implies $T^*_{11} \geq T^*_{22}$ and vice versa. Let $T_{22} = T^*_{11}$ and thus, from Equation (5), $T_{21} = h^{-1}\{h(T^*_{11}) - c_h/c_f\}$. If $C^*(1) \geq C^*(2)$ or $T^*_{11}\ (=T_{22}) \geq T^*_{22}$, then $T_{21} \geq T^*_{21}$ and, from the fact that $h(T)T-H(T)$ is an increasing function of T,

$$\sum_{i=1}^{2} \{h(T_{2i})\, T_{2i} - H(T_{2i})\}$$
$$\geq \sum_{i=1}^{2}\{h(T^*_{2i})\, T^*_{2i} - H(T^*_{2i})\} \tag{7}$$

where, $T_{22} = T^*_{11}$ and $T_{21} = h^{-1}\{h(T^*_{11}) - c_h/c_f\}$

Substituting Equation (4), $\sum_{i=1}^{Q} \{h(T^*_{Qi})\, T^*_{Qi} - H(T^*_{Qi})\} = (K + c_p Q)/c_f$, into Equation (7), we obtain the condition that stocking policy is required:

$$h(T_{21})\, T_{21} - H(T_{21}) \geq c_p/c_f$$

where, $T_{21} = h^{-1}\{h(T^*_{11}) - c_h/c_f\}$

Thus, if T^*_{11} (replacement interval for the classical model) is known, wether stocking policy is required or not can be decided.

If life distribution is Weibull of the form $F(t) = 1-\exp(-\lambda t^{\beta})$, $T^*_{11} = \{(K+c_p)/c_f \lambda(\beta-1)\}^{1/\beta}$ and the condition for stocking is

$$\{(K+c_p)/c_f \lambda(\beta-1)\}^{(\beta-1)/\beta} - \{c_p/c_f \lambda(\beta-1)\}^{(\beta-1)/\beta} \geq c_h/c_f$$

As might be expected, stocking policy is required when K is large and non-stocking is required when c_h is large.

If stocking is required, we must determine the stocking level. Since the expression for $\partial C(Q, \boldsymbol{T}_Q) / \partial Q$ is not readily found, the problem of finding the overall optimum $(Q^*, \boldsymbol{T}^*_{Q^*})$ is better resolved by finding suboptimal solutions $(Q, \boldsymbol{T}^*_Q)$ from Equations (4), (5) and (6) for various fixed positive integer values of Q. From these, the pair which gives the minimum value for $C(Q, \boldsymbol{T}_Q)$ is selected. As the optimal order quantity is determined by a trade-off between fixed ordering cost K and inventory carrying cost c_h, we may find the smallest integer Q which satisfies $C^*(Q+1) - C^*(Q) \geq 0$.

4. EQUAL REPLACEMENT INTERVAL CASE

If we assume that replacement intervals are equal, the problem is considerably simplified. Suppose that $T_{Qi} = T_Q$ for all i, then Equations (1) and (4) degenerate to

$$C(Q, T_Q) = \{K + c_p Q + c_f Q H(T_Q) + c_h Q(Q-1)T_Q/2\}/QT_Q \tag{8}$$

and

$$h(T_Q)T_Q - H(T_Q) = (K + c_p Q)/c_f Q \tag{9}$$

Substituting the T_Q satisfying Equation (9) into Equation (8), we obtain

$$C^*(Q) = c_f h(T^*_Q) + c_h(Q-1)/2 \tag{10}$$

5. NUMERICAL EXAMPLE

Consider a unit having a Weibull lifetime distribution $F(t) = 1 - \exp(-3t^2)$. Suppose that $K = \$30$, $c_p = \$100$, $c_f = \$5$ and $c_h = \$2$. From Equation (5), $T_{Qi} = T_{QQ} - (Q-i)/15$. Substituting T_{Qi} into Equation (4) and making some algebraic manipulation, we obtain $T_{QQ} = \{2/Q + 20/3 - (Q^2-1)/2700\}^{1/2} + (Q-1)/30$. Substituting T_{QQ} into Equation (6), we obtain $C^*(Q) = 2\{450/Q + 1500 - (Q^2-1)/12\}^{1/2} + (Q-1)$. Since $C^*(2) > C^*(3)$ and $C^*(3) < C^*(4)$, the optimal order quantity $Q^* = 3$ and the optimal replacement intervals are $T^*_{31} = 2.641$, $T^*_{32} = 2.707$, $T^*_{33} = 2.774$ and the optimum cost is $C^*(3) = \$ 83.22$

Likewise, in equal replacement case from Equations (9) and (10), $Q^* = 3$, $T^*_3 = 2.708$ and the optimum cost is \$83.24.

6. COMPARATIVE COST BEHAVIOR

Since the optimal order quantity Q^* is determined by a trade-off between fixed ordering cost K and inventory carrying cost c_h, the expected cost per unit time is computed numerically and plotted as a continuous function of K/c_h. Figure 1 shows the respective costs of the three models (C^*_1 : Barlow and Hunter's, C^*_2 : equal replacement interval case, C^*_3 : unequal replacement interval case) in the case of the preceding example when K is varied. As might be expected, if $K = 0$ the costs of the three models are equal. As the ratio K/c_h increases, cost reduction due to quantity pruchase, $C^*_1 - C^*_2$ or $C^*_1 - C^*_3$, increases. However, the difference between C^*_2 and C^*_3 is very little and it does not be systematically related to the ratio K/c_h. This seems to be due to the following fact. If c_h is large, the optimal order quantity Q^* is small and on the other hand if Q^* is large, c_h must be small. Thus one can save little money by removing the equality assumption on replacement intervals. It may be satisfactory to use equal replacement interval model in practice.

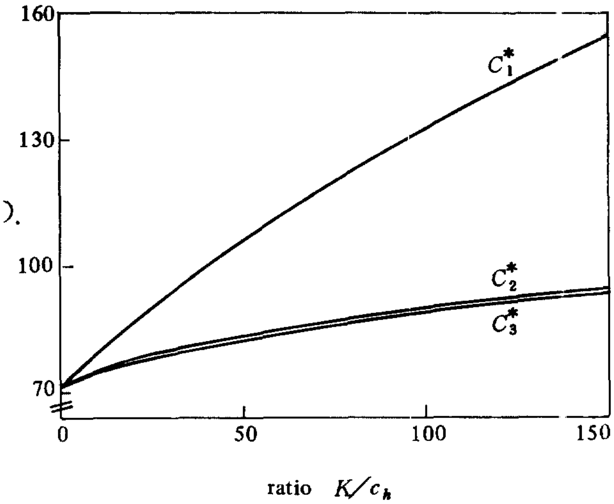

Figure 1. Comparative cost behavior

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고장시 應急修理가 가능한 部品의 政期交換政策을 豫備品의 다량확보가 가능한 경우로 확장하였다. 最適代置期間들은 在庫維持費 때문에 增加數列을 이루었고, 이로부터 最適1회 豫備品 發注量 및 最適 代置期間들을 구할 수 있었다. 各 部品의 代置期間을 같게 두면 문제가 간단해 지면서도 最小費用에는 별 差가 없었다.

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