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FAINT-CONTINUITY AND SET-CONNECTEDNESS

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1. Introduction

In 1971, Jin Ho Kwak [1] introduced a new class of functions called *set-connected* and obtained some sufficient conditions for a set-connected function to be continuous. In 1976, the present author [5] continued the investigation of set-connected functions and showed that every weakly-continuous surjection is set-connected. Quite recently, P. E. Long and L. L. Herrington [3] have introduced a weak form of continuity called faintly-continuous by making use of θ-open sets. They obtained a large number of properties concerning such functions, and among them, showed that every weakly-continuous function is faintly-continuous and faint-continuity is equivalent to almost-continuity in the sense of Singal [10] if the range is almost-regular [9]. The purpose of the present note is to obtain further properties about faint-continuity and set-connectedness. It will be shown that every faintly-continuous surjection is set-connected and the converse is not true even though the range is regular.

2. Preliminaries

Throughout the present note X and Y denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure of S and the interior of S are denoted by $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$, respectively. The subset S is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(S))=S$ (resp. $\mathrm{Cl}(\mathrm{Int}(S))=S$). The set of all $x \varepsilon X$ such that $S \cap \mathrm{Cl}(V) \neq \emptyset$ for every neighborhood V of x is called the *θ-closure* [11] of S. The subset S is called *θ-closed* if the θ-closure of S is contained in S. The complement of a θ-closed set is called θ-open. We shall recall the definitions of some weak forms of continuity. A function $f: X \rightarrow Y$ is said to be *weakly-continuous* [2] (resp. *θ-continuous*, *almost-continuous* [10]) if for each $x \varepsilon X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subset \mathrm{Cl}(V)$ (resp. $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$, $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$).

REMARK 2.1. The following implications are known:

continuity$\Rightarrow$almost-continuity$\Rightarrow\theta$-continuity$\Rightarrow$weak-continuity.

DEFINITION 2.2. A function $f: X \to Y$ is said to be *faintly-continuous* [3] if for each $x \varepsilon X$ and each θ-open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset V$.

THEOREM 2.3. (Long and Herrington [3]). *Every weakly-continuous function is faintly-continuous.*

DEFINITION 2.4. A space X is said to be *connected between A and B* if there exists no closed and open set F of X such that $A \subset F$ and $F \cap B = \emptyset$. A function $f: X \to Y$ is said to be *set-connected* [1] provided that $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to the relative topology if X is connected between A and B.

THEOREM 2.5. (Noiri, [5]). *Every weakly-continuous surjection is set-connected.*

3. Faintly-continuous functions

THEOREM 3.1. *If $f: X \to Y$ is almost-continuous and V is θ-open in Y, then $f^{-1}(V)$ is θ-open in X.*

PROOF. Let V be θ-open in Y and x any point of $f^{-1}(V)$. By Theorem 1 of [3], there exists a regular open set U such that $f(x) \varepsilon U \subset \mathrm{Cl}(U) \subset V$. Thus, we have $x \varepsilon f^{-1}(U) \subset f^{-1}(\mathrm{Cl}(U)) \subset f^{-1}(V)$. Since f is almost-continuous, $f^{-1}(U)$ is open in X and $f^{-1}(\mathrm{Cl}(U))$ is closed in X [10, Theorem 2.2.]. Therefore, we obtain $x \varepsilon f^{-1}(U) \subset \mathrm{Cl}(f^{-1}(U)) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is θ-open in X.

COROLLARY 3.2. (Long and Herrington [3]). *If $f: X \to Y$ is continuous and V is θ-open in Y, then $f^{-1}(V)$ is θ-open in X.*

THEOREM 3.3. *A function $f: X \to Y$ is faintly-continuous if and only if $f^*: X \to Y^*$ is faintly-continuous, where Y^* denotes the semiregularization of Y.*

PROOF. *Necessity.* Let $f: X \to Y$ be faintly-continuous. Let V^* be any θ-open set of Y^*. Since the identity function $i: Y \to Y^*$ is continuous, by Corollary 3.2 $i^{-1}(V^*)$ is θ-open in Y and hence $(f^*)^{-1}(V^*) = (i \circ f)^{-1}(V^*) = f^{-1}(i^{-1}(V^*))$ is open in X.

Sufficiency. Let $f^*: X \to Y^*$ be faintly-continuous. Let V be any θ-open set of Y. Since $i^{-1}: Y^* \to Y$ is almost-continuous, by Theorem 3.1 $i(V)$ is θ-open in Y^* and hence $f^{-1}(V)=(f^*)^{-1}(i(V))$ is open in X.

In Theorem 14 of [3], it is shown that if a function $f: X \to Y$ is weakly-continuous then the graph map $g: X \to X \times Y$ is faintly-continuous. However, it was already known that a function $f: : X \to Y$ is weakly-continuous if and only if the graph map $g: X \to X \times Y$ is weakly-continuous [4, Theorem 1]. Thus, Theorem 14 of [3] is an immediate consequence of Theorem 2.3 and the result stated above.

THEOREM 3.4. *If a surjection $f: X \to Y$ is faintly-continuous, then f is set-connected.*

PROOF. Let V be any open and closed set Y. Then V is θ-open and θ-closed in Y. Since f is faintly-continuous, $f^{-1}(V)$ is open and closed in X by Theorem 9 of [3]. Since f is surjective, it follows from Theorem 2 of [1] that f is set-connected.

A space X is said to be *almost-regular* [9] if for each regular closed set F and each $x \notin F$, there exist disjoint open sets U and V of X such that $F \subset U$ and $x \in V$. It is known that every faintly-continuous function into an almost-regular space is almost-continuous [3, Theorem 11].

REMARK 3.5. Every set-connected surjection is not always faintly-continuous even though the range is a regular space as the following example shows.

EXAMPLE 3.6. Let $I=[0,1]$ be the unit interval, τ the co-countable topology for I and σ the usual topology for I. Let $i:(I,\tau) \to (I,\sigma)$ be the identity function. Then, since (I, σ) is regular and $A=[0,1/2) \in \sigma$, A is θ-open in (I, σ) but $i^{-1}(A) \notin \tau$. Thus, i is a set-connected function without being faintly-continuous.

COROLLARY 3.7. *Connectedness is preserved under faintly-continuous surjections.*

PROOF. This follows from [1, Lemma 4] and Theorem 3.4.

4. Set-connected functions

A space X is said to be *extremally disconnected* if $\mathrm{Cl}(V)$ is open in X for every open set V of X. A function $f: X \to Y$ is said to be *δ-continuous* [6] if for each $x \in X$ and each open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(\mathrm{Int}(\mathrm{Cl}(U)))\subset \mathrm{Int}(\mathrm{Cl}(V))$. It is known in [6] that δ-continuity implies almost-continuity and is independent of continuity.

THEOREM 4.1. *If $f: X\to Y$ is set-connected and Y is extremally disconnected, then f is δ-continuous.*

PROOF. Let $x \in X$ and V be any open neighborhood of $f(x)$. Since Y is extremally disconnected, $\mathrm{Cl}(V)$ is a closed and open set of Y. Since f is set-connected, it follows from Theorem 2 and Remark of [1] that $f^{-1}(\mathrm{Cl}(V))$ is closed and open in X. Put $U=f^{-1}(\mathrm{Cl}(V))$, then U is an open neighborhood of x and $f(\mathrm{Int}(\mathrm{Cl}(U)))\subset \mathrm{Int}(\mathrm{Cl}(V))$. This shows that f is δ-continuous.

COROLLARY 4.2. *Let Y be an extremally disconnected space. Then, for a surjection $f: X\to Y$, the following are equivalent:*

(a) *f is δ-continuous.*

(b) *f is almost-continuous.*

(c) *f is θ-continuous.*

(d) *f is weakly-continuous.*

(e) *f is faintly-continuous.*

(f) *f is set-connected.*

PROOF. This follows from Remark 2.1, Theorems 2.3, 3.4 and 4.1. It should be noted that the condition "surjective" on f is only used to prove the implication: (e)$\Rightarrow$(f).

A space X is said to be ***locally S-closed*** [7] if each point of X has an open neighborhood which is an S-closed subspace of X.

COROLLARY 4.3. *If $f: X\to Y$ is set-connected and Y is locally S-closed regular, then f is continuous.*

PROOF. Since Y is locally S-closed regular, by Theorem 3.5 of [7] Y is extremally disconnected and hence f is δ-continuous by Theorem 4.1. Moreover, since Y is regular, f is continuous [6, Theorem 4.6].

The graph $G(f)$ of a function $f: X\to Y$ is said to be ***extremely closed*** [3] if for each $(x, y)\notin G(f)$ there exist an open set U containing x and a θ-open set V containing y such that $(U\times V)\cap G(f)=\emptyset$.

THEOREM 4.4. *If $f: X\to Y$ is set-connected and Y is extremally disconnected*

Hausdorff, then $G(f)$ is extremely-closed.

PROOF. Since f is set-connected and Y is extremally disconnected, by Theorem 4.1 f is δ-continuous. Moreover, since Y is Hausdorff, it follows from Theorem 5.2 of [6] that for each $(x, y)\notin G(f)$ there exist regular open sets $U\subset X$ and $V\subset Y$ containing x and y, respectively, such that $f(U)\cap V=\emptyset$. Since Y is extremally disconnected, $V=\mathrm{Int}(\mathrm{Cl}(V))$ is open and closed in Y and hence it is θ-open in Y. Therefore, by Theorem 15 of [3] $G(f)$ is extremely-closed.

COROLLARY 4.5. *If $f: X\to Y$ is set-connected and Y is locally S-closed Hausdorff, then $G(f)$ is extremely-closed.*

PROOF. This follows from the fact that every locally S-closed Hausdorff space is extremally disconnected [7, Theorem 3.2].

A function $f: X\to Y$ is said to be *weakly-open* [8] if for every open set U of X $f(U)\subset\mathrm{Int}(f(\mathrm{Cl}(U)))$.

THEOREM 4.6. *Let $f: X\to Y$ be a set-connected weakly-open surjection and assume that $f^{-1}(y)$ is connected for each $y\in Y$. Then, X is connected if and only if Y is connected.*

PROOF. *Necessity.* This follows from Lemma 4 of [1].

Sufficiency. Assume that X is not connected. There exist disjoint nonempty open sets U_1 and U_2 such that $X=U_1\cup U_2$. Since f is weakly-open and U_1, U_2 are closed and open in X, $f(U_1)$ and $f(U_2)$ are open in Y. Moreover, we have $f(U_1)\neq\emptyset$, $f(U_2)\neq\emptyset$ and $Y=f(U_1)\cup f(U_2)$. Next, we show that $f(U_1)\cap f(U_2)=\emptyset$. Assume that $y\in f(U_1)\cap f(U_2)$. Put $G_j=f^{-1}(y)\cap U_j$ for $j=1, 2$. Then, for $j=1, 2$ G_j is a nonempty open set of the subspace $f^{-1}(y)$. We also have $G_1\cup G2=f^{-1}(y)$ and $G_1\cap G_2=\emptyset$. This contradicts that $f^{-1}(y)$ is connected for each $y\in Y$. Therefore, we obtain $f(U_1)\cap f(U_2)=\emptyset$ and hence Y is not connected.

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