Kyungpook Math. J.
Volume 24, Number 2
December, 1984

ON THE AUXILIARY GEOMETRIC MEAN OF ENTIRE FUNCTIONS

By Satendra K. Vaish and Harvir S. Kasana

1. Introduction

Let

(1.1) $$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

be an entire function. Set

$$M(r)\equiv M(r,f)=\max_{|z|=r}|f(z)|,$$

$$\mu(r)\equiv\mu(r,f)=\max_{n\geq 0}\{|a_n|r^n\}$$

and $$\nu(r)\equiv\nu(r,f)=\max\{n:\mu(r)=|a_n|r^n\}.$$

$M(r)$, $\mu(r)$ and $\nu(r)$ are called respectively the maximum modulus, the maximum term and the rank of the maximum term of $f(z)$ for $|z|=r$.

The concept of (p,q)-order and lower (p,q)-order of $f(z)$ having an index pair (p,q), $(p\geqq 1,\ q\geqq 1,\ p\geqq q)$, was introduced by Juneja, Kapoor and Bajpai [1]. Thus $f(z)$ is said to be of (p,q)-order ρ and lower (p,q)-order λ, if

(1.2) $$\lim_{r\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log^{[p]} M(r)}{\log^{[q]} r}=\begin{matrix}\rho(p,q)\equiv\rho\\ \lambda(p,q)\equiv\lambda\end{matrix}$$

where $\log^{[0]} x=x$ and $\log^{[n]} x=\log(\log^{[n-1]} x)$ for $0<\log^{[n-1]} x<\infty$. For the definition of index-pair etc. (see Juneja et al. 1976).

The geometric mean of $f(z)$ for $|z|=r$ has been defined as [15, p.144]:

(1.3) $$G(r)=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\ d\theta\right\}.$$

The following two geometric means $g_k(r)$ and $g_k^*(r)$ were introduced by Kamthan [10], and, Jain and Chugh [8], respectively

(1.4) $$g_k(r)=\exp\left\{\frac{k+1}{r^{k+1}}\int_0^r x^k\log G(x)\ dx\right\},\quad k\in R_+,$$

(1.5) $$g_k^*(r)=\exp\left\{\frac{k+1}{(\log r)^{k+1}}\int_1^r x^{-1}(\log x)^k\log G(x)\ dx\right\},\quad k\in R_+.$$

A number of properties regarding the growths of $g_k(r)$ with respect to $G(r)$ and other auxiliary functions for an entire function of order $\rho(2,1)$ were obtained in ([3], [4], [9]—[14], [16]—[18] etc.). Also, some authors ([6]—[8], [20], [21] etc.) investigated the growth relations of the geometric mean $g_k^*(r)$ for an entire function of order $\rho(2,2)$.

In the present paper we are introducing a unified geometric mean $J_{k,m}(r)$, $k\in R_+$, which we shall term as Auxiliary Geometric Mean (a. g. m.) of $f(z)$ and is given by

$$J_{k,m}(r)=\exp\left\{\frac{k+1}{(\log^{[m-1]} r)^{k+1}}\int_{r_0}^{r}\frac{(\log^{[m-1]} x)^k \log G(x)}{V_{[m-2]}(x)}dx\right\} \tag{1.6}$$

where $m\in I_+$, $V_{[m]}(r)=\prod_{i=1}^{m}\log^{[i]} r$ and r_0 is a constant depending on m.

Our aim in this paper is to investigate certain growth properties of the a. g. m. with respect to $G(r)$ and $n(r)$ (number of zeros of $f(z)$ in $|z|\leqq r$) for an entire funcion of (p,q)-order $\rho(p,q)$ and lower (p,q)-order $\lambda(p,q)$. The results that we obtained here generalize, improve and combined many of the known results (see e.g. [5], [6], [8], [9], [11]—[14], [18] etc.)

2. Statements of theorems

THEOREM 1. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function having (p,q)-order $\rho(p,q)$ and lower (p,q)-order $\lambda(p,q)$, then*

$$\lim_{r\to\infty}\ {}^{\sup}_{\inf}\ \frac{\log^{[p]} F(r)}{\log^{[q]} r}=\begin{matrix}\rho(p,q)\equiv\rho\\ \lambda(p,q)\equiv\lambda,\end{matrix} \tag{2.1}$$

where $F(r)$ may be replaced by $G(r)$ or $J_{k,m}(r)$.

THEOREM 2. *Let $f(z)$ be an entire function having (p,q)-order ρ and $f(0)\neq 0$, then*

$$\frac{\delta_2}{k+\rho+1}\leqq\liminf_{r\to\infty}\frac{\log\{G(r)/J_{k,m}(r)\}}{(\log^{[q-1]} r)^{\rho}} \tag{2.2}$$

$$\leqq\limsup_{r\to\infty}\frac{\log\{G(r)/J_{k,m}(r)\}}{(\log^{[q-1]} r)^{\rho}}\leqq\frac{\delta_1}{k+\rho+1},$$

where,

$$\lim_{r\to\infty}\ {}^{\sup}_{\inf}\ \frac{n(r)V_{[m-1]}(r)}{r(\log^{[m-1]} r)^{\rho-1}}=\begin{matrix}\delta_1\\ \delta_2\end{matrix};\ \delta_1,\ \delta_2\in R_+\cup\{0\}. \tag{2.3}$$

THEOREM 3. *For a class of entire functions for which*

$$\lim_{r\to\infty} \frac{\log^{[2]} J_{k,m}(r)}{\log^{[m]} r} = +\infty, \tag{2.4}$$

we have

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[3]} J_{k,m}(r)}{\log^{[m]} r} = \frac{\log L}{\log l}, \tag{2.5}$$

where,

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\frac{\log G(r)}{\log J_{k,m}(r)}\right\}^{1/\log^{[m]} r} = \frac{L}{l}. \tag{2.6}$$

THEOREM 4. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be an entire function having* (p, q)*-order* ρ, *lower* (p, q)*-order* λ *and* $f(0)\neq 0$, *then*

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]} (n(r) \log r)}{\log^{[q]} r} = \frac{\rho}{\lambda}, \tag{2.7}$$

where $n(r)$ *represents the number of zeros of* $f(z)$ *in* $|z| \leqq r$.

THEOREM 5. *For an entire function* $f(z)$ *of* (p, q)*-order* ρ, *lower* (p,q)*-order* λ, $f(0)\neq 0$ *and* $N(r)=\int_0^r n(x)/x\, dx$, *we find*

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]} N(r)}{\log^{[q]} r} = \frac{\rho}{\lambda}. \tag{2.8}$$

THEOREM 6. *For* $r_2 > r_1 > 0$,

$$\begin{aligned} &\{(\log^{[m-1]} r_2)^{k+1} - (\log^{[m-1]} r_1)^{k+1}\} \log G(r_1) \\ &\leqq (\log^{[m-1]} r_2)^{k+1} \log J_{k,m}(r\ ) - (\log^{[m-1]} r_1)^{k+1} \log J_{k,m}(r_1) \leqq \\ &\{(\log^{[m-1]} r_2)^{k+1} - (\log^{[m-1]} r_1)^{k+1}\} \log G(r_2). \end{aligned} \tag{2.9}$$

THEOREM 7. *Let* $f_1(z)$ *and* $f_2(z)$ *be two entire functions of* (p, q)*-orders* ρ_1, ρ_2 *and lower* (p, q)*-orders* λ_1, λ_2, *respectively and* $f(z)$ *be an entire function satisfying*

$$\log^{[p-1]} F(r, f) \sim [\{\log^{[p-1]} F(r, f_1)\}^{\alpha} \{\log^{[p-1]} F(r, f_2)\}^{\beta}], \quad \alpha, \beta \in R_+ \tag{2.10}$$

Then the (p, q)*-order* ρ *and lower* (p, q)*-order* λ *of* $f(z)$ *are bounded by*

$$\alpha\lambda_1 + \beta\lambda_2 \leqq \lambda \leqq \rho \leqq \alpha\rho_1 + \beta\rho_2, \tag{2.11}$$

and if,

$$\log^{[p]} F(r, f) \sim \{\log^{[p]} F(r, f_1)\}^{\gamma} \{\log^{[p]} F(r, f_2)\}^{1-\gamma}, \; r \in (0, 1) \tag{2.12}$$

then

$$\lambda_1^{\gamma} \lambda_2^{1-\gamma} \leqq \lambda \leqq \rho \leqq \rho_1^{\gamma} \; \rho_2^{1-\gamma}. \tag{2.13}$$

THEOREM 8. *For every entire function* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *of* (p, q)*-order* ρ, *we find*

$$\limsup_{r\to\infty} \frac{\log^{[p-1]}\left\{ r\frac{G(r, f^{(1)})}{G(r, f)} \right\}}{\log^{[q]} r} = \Theta \tag{2.14}$$

in the neighbourhood of points where $f(z) > M(r)\,(\nu(r))^{-1/8}$. *Here* $f^{(1)}(z)$ *denotes the first derivative of* $f(z)$ *and*

$$\begin{aligned} \rho = P(\Theta) \equiv P(\Theta, p, q) &= \Theta \quad \text{if } p>q \\ &= 1+\Theta \quad \text{if } p=q=2 \\ &= \max(1, \Theta) \quad \text{if } 3 \leqq p=q < \infty \\ &= \infty \quad \text{if } p=q=\infty, \end{aligned}$$

$$0 \leqq \Theta \leqq \infty.$$

3. Lemmas

In this section we prove a few lemmas which are needed in the sequel.

LEMMA 1. *log* $G(r)$ *is an increasing convex function of log* r, $f(0) \neq 0$.

PROOF. By Jensen's formula, we have

$$\log G(r) = \log|f(0)| + \int_0^r \frac{n(x)}{x}\, dx$$

$$= \log G(r_0) + \int_{r_0^{(1)}}^r \frac{n(x)}{x}\, dx.$$

This gives,

$$\frac{d[\log G(r)]}{d[\log r]} = n(r).$$

The right hand side is a non-decreasing function of r, since $n(r)$ is a non-decreasing function of r and tends to infinity as $r \to \infty$.

LEMMA 2. $(\log^{[m-1]} r)^{k+1} \{\log G(r)\}^2$ *is an increasing convex function of*

(1) r_0 need not be the same at each occurence.

$(\log^{[m-1]} r)^{k+1} \log J_{k,m}(r)$.

PROOF. We have

$$\frac{d[(\log^{[m-1]} r)^{k+1}\{\log G(r)\}^2]}{d[(\log^{[m-1]} r)^{k+1} \log J_{k,m}(r)]} = \frac{\frac{d}{dr}[(\log^{[m-1]} r)^{k+1}\{\log G(r)\}^2]}{\frac{d}{dr}\left[(k+1)\int_{r_0}^{r} \frac{(\log^{[m-1]} x)^k \log G(x)}{V_{[m-2]}(x)} dx\right]}$$

$$= \log G(r) + \frac{2V_{[m-1]}(r)\, G'(r)}{(k+1)\, G(r)},$$

which increases with r for large values of r, since, by lemma 1, $\log G(r)$ is an increasing convex function of $\log r$.

LEMMA 3. *For* $R > r \geqq 0$,

$$G(r) \leqq M(r) \leqq \{G(r)\}^{(R+r)/R-r}. \tag{3.1}$$

PROOF. This can easily be proved with the help of (1.3) and the Poisson-Jensen formula

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|f(Re^{i\phi})|d\phi}{R^2-R\, r\cos(\theta-\phi)+r^2} - \sum_{\mu=1}^{m} \log\left|\frac{R^2-\bar{a}_\mu\, r\, e^{i\theta}}{R(r\, e^{i\theta}-a_\mu)}\right|.$$

LEMMA 4. *For* $R > r > 1$,

$$\log J_{k,m}(r) \leqq \log G(r) \leqq \frac{(\log^{[m-1]} R)^{k+1}}{(\log^{[m-1]} R)^{K+1} - (\log^{[m-1]} r)^{k+1}} \log J_{k,m}(R). \tag{3.2}$$

PROOF. From (1.6), we have

$$\log J_{k,m}(r) = \frac{k+1}{(\log^{[m-1]} r)^{k+1}} \int_{r_0}^{r} \frac{(\log^{[m-1]} x)^k \log G(x)}{V_{[m-2]}(x)} dx \leqq \log G(r). \tag{3.3}$$

Further,

$$\log J_{k,m}(R) \geqq \frac{k+1}{(\log^{[m-1]} R)^{k+1}} \int_r^R \frac{(\log^{[m-1]} x)^k \log G(x)}{V_{[m-2]}(x)} dx$$

$$\geqq \frac{(k+1)\log G(r)}{(\log^{[m-1]} R)^{k+1}} \int_r^R \frac{(\log^{[m-1]} x)^k}{V_{[m-2]}(x)} dx$$

$$= \frac{(\log^{[m-1]} R)^{k+1} - (\log^{[m-1]} r)^{k+1}}{(\log^{[m-1]} R)^{k+1}} \log G(r).$$

(3.3) and (3.4) complete the proof of lemma 4.

4. Proofs of theorems

THEOREM 1. For $R=Kr,\ K>1$, (3.1) and (3.2) give

$$\log^{[p]} G(r)\leqq\log^{[p]} M(r)\leqq\log^{[p]} G(Kr)+0(1)$$

$$\text{and } \log J_{k,m}(Kr)\geqq\frac{(\log^{[m-1]}K\ r)^{k+1}-(\log^{[m-1]}\ r)^{k+1}}{(\log^{[m-1]}\ Kr)^{k+1}}\ \log G(r)$$

$$=\left[1-\left\{\frac{\log^{[m-1]}\ r}{\log^{[m-1]}Kr}\right\}^{k+1}\right]\log G(r),$$

$$\text{or, } \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log^{[p]}\ G(r)}{\log^{[q]}\ r}=\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log^{[p]}\ M(r)}{\log^{[q]}\ r}=\begin{matrix}\rho\\ \lambda\end{matrix},$$

$$\text{and } \lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log^{[p]}\ J_{k,m}(r)}{\log^{[q]}\ r}\geqq\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log^{[p]}\ G(r)}{\log^{[q]}\ r}=\begin{matrix}\rho\\ \lambda\end{matrix}.$$

Also, from (3.2), we have

$$\lim_{r\to\infty}\begin{matrix}\text{suf}\\ \inf\end{matrix}\ \frac{\log^{[p]}\ J_{k,m}\ (r)}{\log^{[q]}\ r}\leqq\lim_{r\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log^{[p]}\ G(r)}{\log^{[q]}\ r}=\begin{matrix}\rho\\ \lambda\end{matrix}.$$

This completes the proof of theorem 1.

REMARK 1. Theorem 1 is the combination of the following five results investigated by different workers:

(i) *For* $(p, q)=(2,1)$, $F(r)=G(r)$, *the result is due to Srivastava* [18].

(ii) *For* $(p, q)=(2,2)$, $F(r)=G(r)$, *the result is due to Jain and Chugh* [6].

(iii) *For* $(p, q)=(2,1)$, $F(r)=J_{1,1}(r)\equiv g_1(r)\equiv g(r)$, *the result is due to Kamthan* [9].

(iv) *For* $(p,q)=(2,1)$, $F(r)=J_{k,1}(r)\equiv g_k(r)$, *the result is due to Kuldip Kumar* [12].

(v) *For* $(p,q)=(2,2)$, $F(r)=J_{k,2}(r)\equiv g_k^*\ (r)$, *the result is due to Jain and Chugh* [6].

THEOREM 2. Combining (1.3) and (1.6) and using Jensen's formula, we get

$$(4.1)\quad \log\left\{\frac{G(r)}{J_{k,m}(r)}\right\}=\frac{1}{(\log^{[m-1]}\ r)^{k+1}}\int_{r_0}^{r}(\log^{[m-1]}\ x)^{k+1}\ \frac{d}{dx}(\log G(X))dx$$

$$=\frac{1}{(\log^{[m-1]}\ r)^{k+1}}\int_{r_0}^{r}\frac{n(x)}{x}(\log^{[m-1]}\ x)^{k+1}\ dx.$$

From (2.3), we have, for any $\varepsilon>0$ and $r>r_0$,

$$(4.2)\qquad (\delta_2-\varepsilon)\ \frac{r(\log^{[m-1]} r)^{\rho-1}}{V_{[m-2]}(r)} \leqq n(r) \leqq (\delta_1+\varepsilon)\ \frac{r(\log^{[m-1]} r)^{\rho-1}}{V_{[m-2]}(r)}$$

Using right-hand inequality in (4.1), we obtain

$$\log\left\{\frac{G(r)}{J_{k,m}(r)}\right\} < \frac{\delta_1+\varepsilon}{(\log^{[m-1]} r)^{k+1}} \int_{r_0}^{r} \frac{(\log^{[m-1]} x)^{k+\rho}}{V_{[m-2]}(x)}\, dx$$

$$= \frac{(\delta_1+\varepsilon)\,(\log^{[m-1]} r)^{\rho}}{k+\rho+1}\,(1-0(1)),$$

or, $\displaystyle\limsup_{r\to\infty} \frac{\log\{G(r)/J_{k,m}(r)\}}{(\log^{[m-1]} r)^{\rho}} \leqq \frac{\delta_1}{k+\rho+1}.$

Similarly, on using left-hand inequality of (4.2) in (4.1) we find

$$\liminf_{r\to\infty} \frac{\log\{G(r)/J_{k,m}(r)\}}{(\log^{[m-1]} r)^{\rho}} \geqq \frac{\delta_2}{k+\rho+1}.$$

REMARK 2. A result due to Vaish and Srivastava [21], for $(p,q)=(2,2)$, $J_{k,2}(r)\equiv g^*_k(r)$, becomes the particular case of the above theorem.

THEOREM 3. We have

$$\log\,[\log^{[m-1]} r)^{k+1} \log J_{k,m}(r)] = (k+1) \int_{r_0}^{r} \frac{\log G(x)}{\log J_{k,m}(x)}\ \frac{dx}{V_{[m-1]}(x)},$$

since numerator on the right-hand side is the differential coefficient of denominator. This gives

$$\log[(\log^{[m-1]} r)^{k+1} \log J_{k,m}(r)] < (k+1) \int_{r_0}^{r} (L+\varepsilon)^{\log^{[m]} x}\ \frac{dx}{V_{[m-1]}(x)}$$

for any $\varepsilon>0$ and $r>r_0=r_0(\varepsilon)$.

Hence we obtain

$$\log\,[(\log^{[m-1]} r)^{k+1} \log J_{k,m}(r)] < (k+1)\ \frac{(L+\varepsilon)^{\log^{[m]} r}}{\log(L+\varepsilon)}.$$

or, $$\limsup_{r\to\infty} \frac{\log^{[3]} J_{k,m}(r)}{\log^{[m]} r} \leq \log L,$$

since, $$\lim_{r\to\infty} \frac{\log^{[2]} J_{k,m}(r)}{\log^{[m]} r} = +\infty.$$

Further, using lemma 2, we have

$$\log \{(\log^{[m-1]}(2r))^{k+1} \log J_{k,m}(2r)\}$$

$$\geqq (k+1)\int_r^{2r} \frac{(\log G(x))^2}{\log J_{k,m}(x) \log G(x)} \frac{dx}{V_{[m-1]}(x)}$$

$$\geqq (k+1) \frac{(\log G(r))^2}{\log J_{k,m}(r)} \int_r^{2r} \frac{dx}{\log G(x) V_{[m-1]}(x)}$$

$$\geqq (k+1) \frac{(\log G(r))^2}{\log J_{k,m}(r)} \frac{1}{\log G(2r)} \{\log^{[m]}(2r) - \log^{[m]} r\}$$

$$> (k+1)(L-\varepsilon)^{\log^{[m]} r} \frac{\log G(r)}{\log G(2r)} \{\log^{[m]} 2r - \log^{[m]} r\},$$

for a sequence of values of r tending to infinity. Consequently,

$$\limsup_{r\to\infty} \frac{\log^{[3]} J_{k,m}(r)}{\log^{[m]} r} \geqq \log L.$$

In a similar manner, we prove that

$$\liminf_{r\to\infty} \frac{\log^{[3]} J_{k,m}(r)}{\log^{[m]} r} = \log l$$

This proves theorem 3.

REMARK 3. Theorem 3 is the generalization of the following results:

(i) *(see theorem* 11, p. 107, [9]) *due to Kamthan for* $(p,q)=(2,1)$, $J_{1,1}(r)\equiv g_1(r)\equiv g(r)$.

(ii) *(see theorem* 6, p. 254, [11]) *due to Kamthan and Jain for* $(p,q)=(2,1)$, $J_{k,1}(r)\equiv g_k(r)$.

(iii) *(see theorem* 4, p. 44, [12]) *due to Kuldip Kumar for* $(p,q)=(2,1)$, $J_{k,1}(r)\equiv g_k(r)$.

(iv) *(see theorem* 4, p. 124, [8]) *due to Jain and Chugh for* $(p,q)=(2,2)$, $J_{k,2}(r)\equiv g_k^*(r)$.

In (ii) Kamthan and Jain used the hypothesis 'log log $G(r)$ is an increasing convex function of log$r$' instead of (2.4) for proving the result (2.5) for(p,q) $=(2,1)$ and $J_{k,1}(r)\equiv g_k(r)$.

Similary, in (iv) Jain and Chugh used the hypothesis 'log log G(r) is an increasing convex function of log log r' instead of (2.4) for getting the result (2.5) for $(p,q)=(2,2)$ and $J_{k,2}(r)\equiv g_k^*(r)$.

THEOREM 4. From (3.3), we have

$$\log J_{k,m}(r) \leqq \log G(r) = \int_0^r \frac{n(x)}{x} dx + \log|f(0)|$$

$$= \log G(r_0) + \int_{r0}^r \frac{n(x)}{x} dx \leqq n(r) \log r + 0(1),$$

or, $$\lim_{r\to\infty} {\sup \atop \inf} \frac{\log^{[p-1]}(n(r)\log r)}{\log^{[q]} r} \geqq \lim_{r\to\infty} {\sup \atop \inf} \frac{\log^{[p]} J_{k,m}(r)}{\log^{[q]} r} = {\rho \atop \lambda}.$$

Again, we have

$$\log J_{k,m}(r^{\delta}) \geqq \frac{k+1}{(\log^{[m-1]} r^{\delta})^{k+1}} \int_{r^{\delta'}}^{r^{\delta}} \frac{(\log^{[m-1]} x)^k \log G(x)}{V_{[m-2]}(x)} dx, \quad \delta > \delta' > 1$$

$$\geqq \frac{(k+1) \log G(r^{\delta'})}{(\log^{[m-1]} r^{\delta})^{k+1}} \int_{r^{\delta'}}^{r^{\delta}} \frac{(\log^{[m-1]} x)^k}{V_{[m-2]}(x)} dx$$

$$= \log G(r^{\delta'}) \left[1 - \left\{ \frac{\log^{[m-1]} r^{\delta'}}{\log^{[m-1]} r^{\delta}} \right\}^{k-1} \right]$$

$$= \log G(r^{\delta'}) \{1 - 0(1)\}$$

$$> \int_r^{r^{\delta'}} \frac{n(x)}{x} dx \geqq n(r) \log r.$$

This gives,

$$\lim_{r\to\infty} {\sup \atop \inf} \frac{\log^{[p-1]}(n(r)\log r)}{\log^{[q]} r} \leqq \lim_{r\to\infty} {\sup \atop \inf} \frac{\log^{[p]} J_{k,m}(r)}{\log^{[q]} r} = {\rho \atop \lambda}.$$

REMARK 4. For entire functions of non-integral order this theorem gives the following results as particular cases:

(i) *For* $(p,q)=(2,1)$, *the result is given by Boas* [1, p. 15].

(ii) *For* $(p,q)=(2,2)$, *the result is given by Jain and Chugh* [5, p. 98].

THEOREM 5. We have

$$N(r^2) \geqq \int_r^{r^2} \frac{n(x)}{x} dx \geqq n(r) \log r,$$

and $$N(r) = 0(1) + \int_{r_0}^r \frac{n(x)}{x} dx \leqq n(r) \log r \ (1 + 0(1)).$$

Now this theorem follows from theorem 4 and the above two inequalities.

REMARKS 5. The proof of theorem 5 is given by Jain and Chugh [5, p. 99] for $(p,q)=(2,2)$.

THEOREM 6. Since $G(r)$ is an increasing function of r, we have, from (1, 6)

$$(\log^{[m-1]} r_2)^{k+1} \log J_{k,m}(r_2) - (\log^{[m-1]} r_1)^{k+1} \log J_{k,m}(r_1)$$

$$=(k+1)\int_{r_1}^{r_2} \frac{(\log^{[m-1]} x)^k \log G(x)}{V_{[m-2]}(x)}\,dx$$

$$\leqq \{(\log^{[m-1]} r_2)^{k+1} - (\log^{[m-1]} r_1)^{k+1}\} \log G(r_2),$$

and

$$(\log^{[m-1]} r_2)^{k+1} \log J_{k,m}(r_2) - (\log^{[m-1]} r_1)^{k+1} \log J_{k,m}(r_1)$$

$$=(k+1)\int_{r_1}^{r_2} \frac{(\log^{[m-1]} x)^k \log G(x)}{V_{[m-2]}(x)}\,dx$$

$$\geqq \{(\log^{[m-1]} r_2)^{k+1} - (\log^{[m-1]} r_1)^{k+1}\} \log G(r_1).$$

COROLLARY 1. If η $(0<\eta<1)$ is a constant, then

$$\lim_{r\to\infty} \frac{\{J_{k,m}(\exp^{[m-1]}(\eta r))\}^{\eta^{k+1}}}{J_{k,m}(\exp^{[m-1]} r)} = 0.$$

Putting $r_1=\exp^{[m-1]} r$, $r_2=\exp^{[m-1]}(\eta r)$, $(\exp^{[m]} x=\exp(\exp^{[m-1]} x))$, $\exp^{[0]} x =x)$, in (2.9), we get

$$\log G(\exp^{[m-1]} \eta r) \leqq (1-\eta^{k+1})^{-1} \log\left[\frac{J_{k,m}(\exp^{[m-1]} r)}{\{J_{k,m}(\exp^{[m-1]} \eta r)\}^{\eta^{k+1}}}\right] \leqq \log G(\exp^{[m-1]} r)$$

Now, proceeding to limits the result follows.

THEOREM 7. For any $\varepsilon>0$, we have

$$\frac{\log[\log^{[p-1]} F(r,f_1)]^{\alpha}}{\log^{[q]} r} < \alpha\left(\rho_1+\frac{\varepsilon}{2}\right),\quad r>r_1(\varepsilon)$$

$$\frac{\log[\log^{[p-1]} F(r,f_2)]^{\beta}}{\log^{[q]} r} < \beta\left(\rho_2+\frac{\varepsilon}{2}\right),\quad r>r_2(\varepsilon).$$

Adding above two inequalities, we get, for $r>r_0=\max(r_1, r_2)$,

$$\frac{\log[\{\log^{[p-1]} F(r,f_1)\}^{\alpha}\{\log^{[p-1]} F(r,f_2)\}^{\beta}]}{\log^{[q]} r} < \alpha\,\rho_1+\beta\,\rho_2+\frac{1}{2}(\alpha+\beta)\varepsilon.$$

Similarly, for limit infimum as given by (2.1), we find

$$\frac{\log[\{\log^{[p-1]} F(r,f_1)\}^{\alpha}\{\log^{[p-1]} F(r,f_2)\}^{\beta}]}{\log^{[q]} r} > \alpha\,\lambda_1+\beta\,\lambda_2+\frac{1}{2}(\alpha+\beta)\varepsilon.$$

Now, using (2.10), we have for sufficiently large values of r,

$$\alpha\,\lambda_1+\beta\,\lambda_2+\frac{1}{2}(\alpha+\beta)\varepsilon < \frac{\log^{[p]} F(r,f)}{\log^{[q]} r} < \alpha\,\rho_1+\beta\,\rho_2+\frac{1}{2}(\alpha+\beta)\varepsilon,$$

or,
$$\alpha\,\lambda_1+\beta\,\lambda_2 \leqq \lambda \leqq \rho \leqq \alpha\rho_1+\beta\,\rho_2.$$

Again, for any $\varepsilon>0$, we have, from (2.1)

$$\left\{\frac{\log^{[p]} F(r,f_1)}{\log^{[q]} r}\right\}^{\gamma} < (\rho_1+\varepsilon)^{\gamma},\ r>r'(\varepsilon)$$

$$\text{and } \left\{\frac{\log^{[p]} F(r,f_2)}{\log^{[q]} r}\right\}^{1-\gamma} < (\rho_2+\varepsilon)^{1-\gamma},\ r>r''(\varepsilon).$$

On multiplying the above two inequalities, we have, for any $\varepsilon>0$ and $r>r_0=\max(r', r'')$

$$\frac{\{\log^{[p]} F(r,f_1)\}^{\gamma}\{\log^{[p]} F(r,f_2)^{1-\gamma}\}}{\log^{[q]} r} < \rho_1^{\gamma}\,\rho_2^{1-\gamma}\left\{1+\varepsilon\frac{\gamma}{\rho_1}+\frac{1-\gamma}{\rho_2}+0(1)\right\}.$$

Similary, proceeding to limit infimum,

$$\frac{\{\log^{[p]} F(r,f_1)\}^{\gamma}\{\log^{[p]} F(r,f_2)\}^{1-\gamma}}{\log^{[q]} r} > \lambda_1^{\gamma}\,\lambda_2^{1-\gamma}\left\{1-\varepsilon\left(\frac{\gamma}{\lambda_1}+\frac{1-\gamma}{\lambda_2}\right)+0(1)\right\}.$$

On account of (2.12), we find for sufficiently large values of r,

$$\lambda_1^{\gamma}\,\lambda_2^{1-\gamma}\left\{1-\varepsilon\left(\frac{\gamma}{\lambda_1}+\frac{1-\gamma}{\lambda_2}+0(1)\right\}\right. < \frac{\log^{[p]} F(r,f)}{\log^{[q]} r} < \rho_1^{\gamma}\,\rho_2^{1-\gamma}\left\{1-\varepsilon\left(\frac{\gamma}{\rho_1}+\frac{1-\gamma}{\rho_2}+0(1)\right\}\right.$$

Now, taking limits as $r\longrightarrow\infty$, we get (2.13).

COROLLARY 2. Let $f_i(z)$, $i=1, 2, \cdots, n$ be n entire functions of (p,q)-orders ρ_i and lower (p, q)-orders λ_i and $f(z)$ be an entire function satisfying

(4.3) $$\log^{[p-1]} F(r, f) \sim \prod_{i=1}^{n} \{\log^{[p-1]} F(r, f_i)\}^{\alpha_i}, \ \alpha_i \in R_+$$

then (p, q)-order ρ and lower (p, q)-order λ of $f(z)$ are bounded by

(4-4) $$\sum_{i=1}^{n} \alpha_i \lambda_i \leqq \lambda \leqq \rho \leqq \sum_{i=1}^{n} \alpha_i \rho_i,$$

and if

(4.5) $$\log^{[p]} F(r, f) \sim \prod_{i=1}^{n} \{\log^{[p]} F(r, f_i)\}^{\gamma_i}, \ \gamma_i \in (0, 1), \ \sum_{i=1}^{n} \gamma_i = 1$$

then,

(4.6) $$\prod_{i=1}^{n} \lambda_i^{\gamma_i} \leqq \lambda \leqq \rho \leqq \prod_{i=1}^{n} \rho_i^{\gamma_i}.$$

This corollary is an immediate generalization of the above theorem to the case of n entire functions.

COROLLARY 3. Let $f_1(z)$ and $f_2(z)$ be two entire functions of regular (p, q) growth. Then $f(z)$ is also of regular (p, q) growth and its order is given by $\rho = \alpha\rho_1 + \beta\rho_2$ and $\rho = \rho_1^{\gamma} \rho_2^{1-\gamma}$ under the conditions (2.10) and (2.12), respectively.

THEOREM 8. We have

$$G(r, f^{(1)}) = \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \log|f^{(1)}(r e^{i\theta})| d\theta\right\}$$

$$= \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \log \left|\frac{f^{(1)}(r e^{i\theta})}{f(r e^{i\theta})}\right| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta\right\}.$$

Also, we have [19, p. 103], in the neighbourhood of points, where $|f(z)| > M(r)(\nu(r))^{-1/8}$,

$$\frac{f'(z)}{f(z)} = \{1 + h(z)\ (\nu(R))^{-1/16}\}\ \frac{\nu(r)}{z}, \quad |h| < k$$

where $\nu(r)$ denotes the rank of the maximum term in $f(z)$, for $|z| = r$.

Hence, in the neighbourhood of points, where $|f(z)| > M(r)(\nu(r))^{-1/8}$,

$$G(r, f^{(1)}) = G(r, f) \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \log(|1 + h(z)\ (\nu(R))^{-1/16}| \left|\frac{\nu(r)}{z}\right|)\ d\theta\right\}$$

(4.7) $$> G(r, f)\ \frac{\nu(r)}{r}\ (1 - k(\nu(R))^{-1/16})$$

and

(4.8) $$G(r, f^{(1)})<G(r, f)\ \frac{\nu(r)}{r}\ (1+k(\nu(R))^{-1/16}).$$

Proceeding to limits, as $r\longrightarrow\infty$, (4.7) and (4.8) give

$$\limsup_{r\to\infty}\frac{\log^{[p-1]}\left\{r\frac{G(r,f^{(1)})}{G(r,f)}\right\}}{\log^{[q]} r}=\limsup_{r\to\infty}\ \frac{\log^{[p-1]}\ (r)}{\log^{[q]}\ r}=\Theta.$$

This completes the proof of theorem 8.

COROLLARY 5. For an entire function $f(z)=\sum_{n=0}^{\infty} a_n\ z^n$ of (p,q)-order ρ,

(4.9) $$\limsup_{r\to\infty}\frac{\log^{[p-1]}\left[r\left\{\frac{G(r,f^{(n)})}{G(r,f)}\right\}^{1/n}\right]}{\log^{[q]}\ r}=\Theta.$$

From (4.7) and (4.8), we have

$$G(r, f^{(s)})>G(r, f^{(s-1)})\ \frac{\nu(r)}{r}(1-k\,(\nu(R))^{-1/16})$$

and $G(r, f^{(s)})<G(r, f^{(s-1)})\ \frac{\nu(r)}{r}(1+k\,(\nu(R))^{-1/16})$.

Taking $s=1, 2, \cdots, n$ and multiplying all the inequalities thus obtained and proceeding to limits (4.9) follows.

Department of Mathematics,
University of Roorkee,
Roorkee-247672 (U.P) India

BIBLIOGRAPHY

[1] R.P. Boas, *Entire Functions*, Acad. Press, New York(1954).

[2] O.P. Juneja, G.P. Kapoor and S.K. Bajpai, *On the (p,q)-order and lower (p,q)-order of an entire function*, J. reine angew. Math., **282**(1976), 53—67.

[3] P.K. Jain, *On the mean values of an entire function*, Math. Nachr., **44**(1970), 305—312.

[4] ________, *Growth of geometric means of an entire function*, J. Math. Sci., 7(1972), 78—85.

[5] P.K. Jain and V.D. Chugh, *On the logarithmic convergence exponent of the zeros of entire functions*, The Yokohama Math. J., No.2, **21**(1973), 97—101.

[6] ________, *Sur les zéros et moyennes d'une classe de fonctions entires*, Bull. Sc.

Math., 2e serie, **98**(1974), 17—28.

[7] ________, *Logarithmic proximate order and geometric means of an entire function of order zero,* Riv. Mat. Univ. Parma, (4)**2**(1976), 57—68.

[8] ________, *The geometric means of an entire function of order zero,* Collect. Math., **28**(1977), 115—129.

[9] P.K. Kamthan, *On the mean values of an entire function,* The Mathematics Student, **32**(1964), 101—109.

[10] ________, *On the mean values of an entire function* (IV), Math. Japon., **12**(1968), 121—129.

[11] P.K. Kamthan and P.K. Jain, *The geometric means of an entire function,* Ann. Polon. Math., **21**(1969), 247—255.

[12] Kuldip Kumar, *On the geometric means of an integral function,* Monatsch. für Math., 70(1966), 39—46.

[13] ________, *On the geometric means of an integral function,* Compositio Math., **16** (1968), 271—277.

[14] ________, *A note on the geometric means of an integral function,* Ganita, **27**(1976), 57—61.

[15] G. Polya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis* I, Ver Gottingenm Berlin (1925).

[16] S.M. Shah, *A note on means of entire functions,* Publ. Math. Debrecen, **2**(1951), 95—99.

[17] J.P. Singh, *Growth, of geometric means of an entire function,* Publ. Math. Debrecen, **21**(1974), 161—169.

[18] S.N. Srivastava, *On the geometric mean of an integral function,* Ganita, **13**(1962), 87—98.

[19] G. Valiron, *Lectures on General Theory of integral functions,* Chelsea Publishing Co., New York(1949).

[20] S.K. Vaish, *Geometric means of an entire function of order zero,* Rev. Roum. Math. Pures et Appl., no.9, **23**(1978), 1425—1428.

[21] S.K. Vaish and G.S. Srivastava, *On the geometric means of an entire function of slow growth,* Jñānabh, **11**(1981), 5—13.