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ON CERTAIN CLASSES OF UNIVALENT FUNCTIONS IN THE UNIT DISK

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1. Introduction

Let A denote the class of functions

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

analytic in the unit disk $U=\{|z|<1\}$. A function $f(z)\in A$ is said to be univalent and starlike if, and only if,

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0$$

for $z\in U$. Recently, the general idea of order for a starlike function has been introduced in a number of ways as in M. S. Robertson [8], R. J. Libera [2], K. S. Padmanabhan [7] and F. Holland and D. K. Thomas [1]. According to K. S. Padmanabhan [7], a function $f(z)\in A$ is said to be starlike of order k in the unit disk U if the condition

$$\left|\frac{\dfrac{zf'(z)}{f(z)}-1}{\dfrac{zf'(z)}{f(z)}+1}\right|<k$$

hold for some $k(0<k\leqq 1)$ and $z\in U$. The class of such functions we denote by $S(k)$. And let $C(k)$ denote the class of functions $f(z)\in A$ such that $zf'(z)$ is in the class $S(k)$.

For the class $S(k)$, K. S. Padmanabhan [7] has given representation formula, some distortion theorems and the radius of convexity. Moreover, M. L. Mogra [3] has shown a sufficient condition for a function in the class $S(k)$.

2. The necessary and sufficient conditions

THEOREM 1. *A function*

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

is in the class $S(k)$ if, and only if,

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$$\sum_{n=2}^{\infty}\{(n-1)+k(n+1)\}a_n \leqq 2k.$$

The equality holds for the function

$$f(z)=z-\sum_{n=2}^{\infty}\frac{2k}{(n-1)+k(n+1)}z^n.$$

PROOF. Assume that

$$\sum_{n=2}^{\infty}\{(n-1)+k(n+1)\}a_n \leqq 2k$$

and let $|z|=1$. Then we have

$$\begin{aligned}
&|zf'(z)-f(z)|-k|zf'(z)+f(z)| \\
&=|\sum_{n=2}^{\infty}(1-n)a_n z^n|-k|2z-\sum_{n=2}^{\infty}(n+1)a_n z^n| \\
&\leqq |z|[\sum_{n=2}^{\infty}\{(n-1)+k(n+1)\}a_n-2k] \\
&\leqq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z)\in S(k)$.

For the converse, assume that

$$\left|\frac{\dfrac{zf'(z)}{f(z)}-1}{\dfrac{zf'(z)}{f(z)}+1}\right|=\left|\frac{\sum_{n=2}^{\infty}(n-1)a_n z^n}{2z-\sum_{n=2}^{\infty}(n+1)a_n z^n}\right| < k.$$

Since $|\mathrm{Re}(z)|\leqq|z|$ for any z, we have

$$Re\left\{\frac{\sum_{n=2}^{\infty}(n-1)a_n z^n}{2z-\sum_{n=2}^{\infty}(n+1)a_n z^n}\right\}<k. \tag{1}$$

Choose values of z on the real axis so that $zf'(z)/f(z)$ is real. Upon clearing the denominator in (1) and letting $z\rightarrow 1$ through real values, we obtain

$$\sum_{n=2}^{\infty}(n-1)a_n \leqq k\{2-\sum_{n=2}^{\infty}(n+1)a_n\}$$

This inequality gives the required condition. Furthermore, the function

$$f(z)=z-\sum_{n=2}^{\infty}\frac{2k}{(n-1)+k(n+1)}z^n$$

is an extremal function for the theorem.

THEOREM 2. *A function*

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n \geqq 0)$$

is in the class $C(k)$ *if, and only if,*

$$\sum_{n=2}^{\infty}n\{(n-1)+k(n+1)\}a_n \leqq 2k.$$

The equality holds for the function

$$f(z)=z-\sum_{n=2}^{\infty}\frac{2k}{n\{(n-1)+k(n+1)\}}z^n$$

The proof of Theorem 2 is obtained by using the same technique as in the proof of Theorem 1.

3. Some properties for the classes $S(k)$ and $C(k)$

THEOREM 3. *Let* $0<k_1 \leqq k_2 \leqq 1$. *Then we have*

$$S(k_1) \supset S(k_2).$$

PROOF. Let a function

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n \geqq 0)$$

be in the class $S(k_2)$ and $k_1=k_2-\delta$. Then, by Theorem 1, we have

$$\sum_{n=2}^{\infty}\{(n-1)+k_2(n+1)\}a_n \leqq 2k_2$$

and

$$\sum_{n=2}^{\infty}a_n \leqq \frac{2k_2}{1+3k_2}<1.$$

Consequently, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\{(n-1)+k_1(n+1)\}a_n \\
&=\sum_{n=2}^{\infty}\{(n-1)+(k_2-\delta)(n+1)\}a_n \\
&\leqq \sum_{n=2}^{\infty}\{(n-1)+k_2(n+1)\}a_n-3\delta\sum_{n=2}^{\infty}a_n \\
&\leqq 2k_2-3\delta \\
&\leqq 2k_1.
\end{aligned}$$

This completes the proof of the theorem with the aid of Theorem 1.

THEOREM 4. *Let* $0<k_1\leqq k_2\leqq 1$. *Then we have*

$$C(k_1)\supset C(k_2).$$

The proof of Theorem 4 is given in much the same way as Theorem 3 with the aid of Theorem 2.

4. Distortion theorems for the fractional calculus

There are many definitions of the fractional calculus. In 1978, S. Owa [6] gave the following definitions for the fractional calculus.

DEFINITION 1. The fractional integral of order α is defined by

$$D_z^{-\alpha}f(z)=\frac{1}{\Gamma(\alpha)}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\alpha}},$$

where $\alpha>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$. Moreover,

$$f(z)=\lim_{\alpha\to 0}D_z^{-\alpha}f(z).$$

DEFINITION 2. The fractional derivative of order α is defined by

$$D_z^{\alpha}f(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{\alpha}},$$

where $0<\alpha<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$. Moreover,

$$f(z)=\lim_{\alpha\to 0}D_z^{\alpha}f(z).$$

DEFINITION 3. Under the conditions of Definition 2, the fractional derivative of order $(n+\alpha)$ is defined by

$$D_z^{n+\alpha}f(z)=\frac{d^n}{dz^n}D_z^{\alpha}f(z),$$

where $n\in N\cup\{0\}$.

For other definitions of the fractional calculus, see T. J. Osler [5], *B.* Ross [9], K. Nishimoto [4] and M. Saigo [10].

LEMMA 1. *Let a function*

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n \geqq 0)$$

be in the class $S(k)$. Then we have

$$|z|-\frac{2k}{1+3k}|z|^2 \leqq |f(z)| \leqq |z|+\frac{2k}{1+3k}|z|^2$$

and

$$1-\frac{4k}{1+3k}|z| \leqq |f'(z)| \leqq 1+\frac{4k}{1+3k}|z|$$

for $z \in U$. The equalities hold for the function

$$f(z)=z-\frac{2k}{1+3k}z^2.$$

PROOF. By using Theorem 1, we have

$$\sum_{n=2}^{\infty}a_n \leqq \frac{2k}{1+3k}$$

and

$$\sum_{n=2}^{\infty}na_n \leqq \frac{4k}{1+3k}.$$

Hence we have

$$|f(z)| \leqq |z|+|z|^2\sum_{n=2}^{\infty}a_n$$

$$\leqq |z|+\frac{2k}{1+3k}|z|^2,$$

$$|f(z)| \geqq |z|-|z|^2\sum_{n=2}^{\infty}a_n$$

$$\geqq |z|-\frac{2k}{1+3k}|z|^2,$$

$$|f'(z)| \leqq 1+|z|\sum_{n=2}^{\infty}na_n$$

$$\leqq 1+\frac{4k}{1+3k}|z|$$

and

$$|f'(z)| \geqq 1-|z|\sum_{n=2}^{\infty}na_n$$

$$\geqq 1-\frac{4k}{1+3k}|z|$$

for $z \in U$.

LEMMA 2. *Let a function*

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n \geqq 0)$$

be in the class $C(k)$. Then we have

$$|z|-\frac{k}{1+3k}|z|^2 \leqq |f(z)| \leqq |z|+\frac{k}{1+3k}|z|^2$$

and

$$1-\frac{2k}{1+3k}|z| \leqq |f'(z)| \leqq 1+\frac{2k}{1+3k}|z|$$

for $z \in U$. *The equalities hold for the function*

$$f(z)=z-\frac{k}{1+3k}z^2.$$

PROOF. By using Theorem 2, we have

$$\sum_{n=2}^{\infty}a_n \leqq \frac{k}{1+3k}$$

and

$$\sum_{n=2}^{\infty}na_n \leqq \frac{2k}{1+3k}.$$

Therefore, we have

$$|f(z)| \leqq |z|+|z|^2\sum_{n=2}^{\infty}a_n$$

$$\leqq |z|+\frac{k}{1+3k}|z|^2,$$

$$|f(z)| \geqq |z|-|z|^2\sum_{n=2}^{\infty}a_n$$

$$\geqq |z|-\frac{k}{1+3k}|z|^2,$$

$$|f'(z)| \leqq 1+|z|\sum_{n=2}^{\infty}na_n$$

$$\leqq 1+\frac{2k}{1+3k}|z|$$

and

$$|f'(z)| \geqq 1-|z|\sum_{n=2}^{\infty}na_n$$

$$\geqq 1-\frac{2k}{1+3k}|z|$$

for $z \in U$.

THEOREM 5. *Let a function*

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n\geqq 0)$$

be in the class $S(k)$. *Then we have*

$$|D_z^{-\alpha}f(z)|\geqq\frac{|z|^{1+\alpha}(1+3k-2k|z|)}{(1+3k)\Gamma(2+\alpha)},$$

$$|D_z^{-\alpha}f(z)|\leqq\frac{|z|^{1+\alpha}(1+3k+2k|z|)}{(1+3k)\Gamma(2+\alpha)},$$

$$|D_z^{1-\alpha}f(z)|\geqq\frac{|z|^{\alpha}\{(1+3k)(1-\alpha)-2k(2+\alpha)|z|\}}{(1+3k)\Gamma(2+\alpha)}$$

and

$$|D_z^{1-\alpha}f(z)|\leqq\frac{|z|^{\alpha}\{(1+3k)(1+\alpha)+2k(2+\alpha)|z|\}}{(1+3k)\Gamma(2+\alpha)}$$

for $0<\alpha<1$ *and* $z\in U$.

PROOF. Let

$$\begin{aligned}F(z)&=\Gamma(2+\alpha)z^{-\alpha}D_z^{-\alpha}f(z)\\&=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)}a_n z^n.\end{aligned}$$

Then we have

$$\begin{aligned}&\sum_{n=2}^{\infty}\{(n-1)+k(n+1)\}\frac{\Gamma(n+1)\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)}a_n\\&\leqq\sum_{n=2}^{\infty}\{(n-1)+k(n+1)\}a_n\\&\leqq 2k,\end{aligned}$$

because

$$0<\frac{\Gamma(n+1)\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)}<1.$$

Consequently, the function $F(z)$ belongs to the class $S(k)$ by Theorem 1. Hence we have the theorem with the aid of Lemma 1.

COROLLARY 1. *Under the hypotheses of Theorem* 5, $D_z^{-\alpha}f(z)$ *and* $D_z^{1-\alpha}f(z)$ *are included in the disks with center at the origin and radii* $(1+5k)/(1+3k)\Gamma(2+\alpha)$ *and* $(1+\alpha+5k\alpha+7k)/(1+3k)\Gamma(2+\alpha)$, *respectively.*

THEOREM 6. *Let a function*

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n \geqq 0)$$

be in the class $C(k)$. *Then we have*

$$|D_z^{-\alpha}f(z)| \geqq \frac{|z|^{1+\alpha}(1+3k-k|z|)}{(1+3k)\Gamma(2+\alpha)},$$

$$|D_z^{-\alpha}f(z)| \leqq \frac{|z|^{1+\alpha}(1+3k+k|z|)}{(1+3k)\Gamma(2+\alpha)},$$

$$|D_z^{1-\alpha}f(z)| \geqq \frac{|z|^{\alpha}\{(1+3k)(1-\alpha)-k(2+\alpha)|z|\}}{(1+3k)\Gamma(2+\alpha)}$$

and

$$|D_z^{1-\alpha}f(z)| \leqq \frac{|z|^{\alpha}\{(1+3k)(1+\alpha)+k(2+\alpha)|z|\}}{(1+3k)\Gamma(2+\alpha)}$$

for $0<\alpha<1$ *and* $z \in U$.

The proof of Theorem 6 is obtained by using the same technique as in the proof of Theorem 5 with the aid of Lemma 2.

COROLLARY 2. *Under the hypotheses of Theorem* 6, $D_z^{-\alpha}f(z)$ *and* $D_z^{1-\alpha}f(z)$ *are included in the disks with center at the origin and radii* $(1+4k)/(1+3k)\Gamma(2+\alpha)$ *and* $(1+\alpha+4k\alpha+5k)/(1+3k)\Gamma(2+\alpha)$, *respectively.*

THEOREM 7. *Let a function*

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n \geqq 0)$$

be in the class $C(k)$. *Then we have*

$$|D_z^{\alpha}f(z)| \geqq \frac{|z|^{1-\alpha}(1+3k-2k|z|)}{(1+3k)\Gamma(2-\alpha)}$$

and

$$|D_z^{\alpha}f(z)| \leqq \frac{|z|^{1-\alpha}(1+3k+2k|z|)}{(1+3k)\Gamma(2-\alpha)}$$

for $0<\alpha<1$ *and* $z \in U$ *and*

$$|D_z^{1+\alpha}f(z)| \geqq \frac{|z|^{-\alpha}\{(1+3k)(1-\alpha)-2k(2+\alpha)|z|\}}{(1+3k)\Gamma(2-a)}$$

and

$$|D_z^{1+\alpha}f(z)| \leqq \frac{|z|^{-\alpha}\{(1+3k)(1+\alpha)+2k(2+\alpha)|z|\}}{(1+3k)\Gamma(2-\alpha)}$$

for $0<\alpha<1$ and $z\in U-\{0\}$.

PROOF. Let

$$G(z)=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}f(z)$$
$$=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_n z^n.$$

Then we have

$$\sum_{n=2}^{\infty}\{(n-1)+k(n+1)\}\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_n$$
$$\leqq\sum_{n=2}^{\infty}n\{(n-1)+k(n+1)\}a_n$$
$$\leqq 2k,$$

for

$$0<\frac{\Gamma(n+1)(\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}<n$$

and $f(z)\in C(k)$. Accordingly, the function $G(z)$ is in the class $S(k)$ by means of Theorem 1. Therefore, we have the theorem by using Lemma 1.

COROLLARY 5. *Under the hypotheses of Theorem* 7, $D_z^{\alpha}f(z)$ *and* $D_z^{1+\alpha}f(z)$ *are included in the disks with center at the origin and radii* $(1+5k)/(1+3k)\Gamma(2-\alpha)$ *ana* $(1+\alpha+5k\alpha+7k)/(1+3k)\Gamma(2-\alpha)$, *respectively*.

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