Kyungpook Math. J.
Volume 24, Number 1
June, 1984

ON OSCILLATIONS OF PERTURBED SECOND ORDER DIFFERENTIAL EQUATION

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A perturbed second order linear differential equation is considered. A criteria concerning the oscillatory behaviour of solutions of the perturbed equation based on the perturbing function and the solutions of the associated unperturbed equation is given.

1. In this paper we shall consider the linear differential equations

$$(p(t)\ x'(t))'+q(t)\ x(t)=0 \qquad (1)\ ,'=d/dt$$

$$(p(t)\ y'(t))'+q(t)\ y(t)=f(t) \qquad (2)$$

where p, q and f are real-valued continuous functions on $[a, \infty)$ where a is anyreal number, $p(t)>0$ for $t\geqslant 0$, $f(t)\neq 0$ on $[a, \infty)$.

The problem of determining oscillation criteria for second order linear differential equations has received a great deal of attention in the last twenty years-see for example [1], [2], [3], [4], [5].

Before proceeding, we shall require some definitions and lemmas:

DEFINITION 1. A solution of (1) (or (2)) is said to be nonoscillatory on $[a, \infty)$ if it has only a finite number of zeros on $[a_1, \infty)$ for some $a_1>a$ and is oscillatory if it has an infinite number of zeros on $[a_1, \infty)$. Equation (1) (or (2)) is oscillatory if it has at least one oscillatory solution on $[a, \infty]$ and is nonoscillatory if all solutions are nonoscillatory.

It is well known from [4] that if x_1 and x_2 are solution basis for equation (1), then the general solution of (2) is given by:

$$y(t)=C_1x_1(t)+C_2x_2(t)+y_p, \qquad (3)$$

where C_1 and C_2 are arbitrary constants and

$$y_p=\sum_{i=1}^{2}x_i(t)\int_a^t \frac{f(s)W_i(s)}{W(x_1, x_2)(s)}ds,$$

where $W(x_1, x_2)(t)$ is the quasi-Wronskian of the solutions x_1 and x_2 and

$W_i\ (t)$ is the determinant obtained from $W(x_1,\ x_2)(t)$ by replacing the i_{th} column with vector $[0,\ 1]^T$, $i=1,\ 2$, i.e.

$$W(x_1,\ x_2)(t)=\begin{vmatrix} x_1 & x_2 \\ px_1' & px_2' \end{vmatrix}=x_1 px_2'-x_2 px_1'=K,\ \ on\ \ [a,\ \infty).$$

DEFINITION 2. The solution basis x_1 and x_2 of (1) is called a *normalized solution basis* if $K=1$ and W is called a *normalized Wronskian.*

From the above definitions it follows that if x_1 and x_2 is a normalized solution basis for equation (1), then equation (3) takes the form

$$y(t)=\sum_{i=1}^{2}x_i(t)\Big[C_i+\int_a^t f(s)\ W_i(s)ds\Big] \tag{4}$$

LEMMA. *If* $\int_0^\infty [1/p(t)]\,dt=\infty$ *and if all solutions of equation* (1) *are bounded, then* (1) *is oscillatory.* ([4])

Supposition A: Let $q(t)>0$ on $[a,\ \infty)$. If there is a nonoscillatory solution of (2) such that $sgn\ y(t)\neq sgn\ f(t)$ for large t then (1) is nonoscillatory.

LEMMA 1. *If* $x_1(t)$ *and* $x_2\ (t)$ *be a (normalized) solution basis for equation* (1), *then* W_1 *and* W_2 *are a (normalized) solution basis for* (1).

PROOF. The proof is trivial since $W_1=-x_2$ and $W_2=x_1$.

THEOREM 1. *Assume* $q(t)>0$. *If equation* (1) *is oscillatory, then all nonoscillatory solutions of equation* (2) *are of the same sign. Moreover if* $f(t)\neq 0$ *for large* t, *then all nonoscillatory solutions of equation* (2) *are of the same sign as* $f(t)$.

PROOF. Let $y(t)$ and $z(t)$ be any two nonoscillatory solutions of equation (2). Since every solution of euation (1) is oscillatory, by hypotheses, $y-z$ is an oscillatory solution of (1). Assume $sgn\ y\neq sgn\ z$ for large t, then $y-z$ is nonoscillatory solution of (1) for large t, which is a contradiction. Thus $sgn\ y=sgn\ z$. Assume $sgn\ y\neq sgn\ f$, then by supposition A, equation (1) is non-osocillatory, which is again a contradiction. Thus nonoscillatory solutions of (2) are of the same sign as $f(x)$. This completes the proof.

THEOREM 2. *Assume $x(t)$ is an oscillatory solution of* (1) *and $y(t)$ is a nonoscillatory solution of* (2). *Then there exists a sequences $\{t_i\}$, $t_i \to \infty$ as $i \to \infty$ such that $(x\,p\,y' - ypx')(t_i) = 0$ for all i.*

PROOF. The agreement is similar to the proof of Sturm separation theorem and therefore is omitted.

THEOREM 3. *Let all solutions of* (2) *be nonoscillatory and of the same sign. If $x(t)$ solution of* (1) *such that $\liminf_{t\to\infty} p(t) > 0$, then the particular solution y_p of* (2) *has the property that $\lim_{t\to\infty} |y_p(t)| = \infty$.*

PROOF. Let $y_i(t) = x(t) + y_p(t)$ denote any nonoscillatory solution of (2). Since all solutions of (2) are of the same sign on $[a, \infty)$, for each i there exists a point t_i such that $y_i(t) \neq 0$ on $[t_i, \infty)$. Without loss of generality, we can assume that $y_i(t) > 0$ on $[t_i, \infty)$ for all i. Then it is clear that $y_p(t) > -Cx(t)$ on $[t_i, \infty)$, and hence $|y_p(t)| > |x(t)|$, for large t. Since $\liminf |x(t)| > 0$ as $t \to \infty$, it follows that $|y_p(t)| \to \infty$, as $t \to \infty$ and the result follows.

THEOREM 4. *If $\int_0^1 [1/p(t)]\, dt = \infty$ and solutions of* (1) *are bounded, then* (2) *has at most one nonoscillatory solution.*

PROOF. It follows from the variation of parameters (constants) formula and hypotheses, that all solutions of (2) are bounded. Assume that y_1 and y_2 are any solutions of (2), and hence their difference $y_1 - y_2 = x(t)$ is a solution of (1). Thus from equation (2) we obtain.

$$y_2(py_1')' - y_1(py_2')' = (y_1 - y_2)\, f(t) \tag{5}$$

The left hand side of (5) can be written as $[y_1(py_2) - y_2(py_1)]'$ and hence equation (5) takes the form

$$[y_1(py_2') - y_2(py_1')]' = x(t)\, f(t) \tag{6}$$

from hypotheses on $x(t)$ and $f(t)$, it follows that the left hand side of (6) has a limit as $t \to \infty$. To prove that equation (2) has at most one nonoscillatory solution: On the contrary assume that there exists two distinct nonoscillatory solutions y_1 and y_2 of equation (2). Let $x_1 = y_1 - y_2$ and $x_2(t)$ be a normalized solution basis for equation (1) i.e. $W(x_1, x_2)(t) = 1 = (x_1 p x_2' - x_2 p x_1')$. Let $y_3 = y_2 + x_2$ and $z(t) = \tan^{-1}(y_3/y_1)$ then

$$z(t)=[y_1 py_3'-y_3 py_1']/P(y_1^2+y_3^2). \tag{7}$$

Since $y_1(t)$ is nonoscillatory solution of (2), by assumption, there exists $t_1>0$ such that $y_1^2>0$ for $t>t_1$, which means that $z(t)$ and, consequently, $z(t)$ are well defined for all $t>t_1$. The numerator in the right hand side of (7) can be rewritten as

$$[y_1 py_3'-y_3 py_1'] = [x_1 px_2'-x_2 px_1'] + [x_1-x_2]py_2'-y_2 p[x_1-x_2']'$$

Assume $x(t)=x_1-x_2$, noting that x_1 and x_2 are normalized solution baris of (1) and using theorem 2, it follows that $(ypy_3'-y_3 py_1')(t)$ has the limit 1 as $t\to\infty$. If we choose $t_2>t_1$ so that

$$(y_1 py_3'-y_3 py_1')>\frac{1}{3} \text{ for } t>t_2$$

then equation (7) takes the form

$$z(t)>1/3\,p(y_1^2+y_3^2)>0,\ t>t_2$$

Thus $z(t)$ is strictly monotone for $t>t_2$.

In addition, since y_1 and y_3 are bounded, there exists a positive number M such that $y_1^2+y_3^2<M$ and it follows that $z(t)>1/3PM$ and hence

$$z(t)>z(t_1)+\frac{1}{3PM}\int_{t_1}^{t} ds \tag{8}$$

Since the integral on the right of (8) diverges and hence $Z(t)$ becomes unbounded as $t\to\infty$. This implies that both $y_1(t)$ and $y_3(t)$ must oscillate which contradicts that y_1 is nonoscillatory solution of (2).

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