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A NOTE ON THE PERIODICITY OF SEQUENCES OF DERIVED SETS*

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We generalize the Tucker's definition about the periodicity of the consecutive derived sets and we prove that with this definition the period is also not greater than 2. Moreover, it is shown that the period not greater than 1 characterizes the T_{D^2} spaces.

1. DEFINITION. Let λ be an ordinal number, A a subset of a topological space, X and dA the derived set of A. The derived set of order λ of A, $d^{\lambda}A$, is defined by transfinte induction in the following way:

$$d^0A=A,\ d^{\lambda+1}A=dd^{\lambda}A,\ d^{\mu}A=\bigcap_{2\leqq\lambda<\mu} d^{\lambda}A\ (\mu \text{ limit ordinal})$$

2. PROPOSITION(see [2]). *A subset A of a topological space has three associated sets*

$$X_A=\overline{d^{2n+1}A}\setminus d^{2n+1}A=d^{2n}A\setminus\overline{hd^{2n-1}A},\ n\geqq 1$$

$$Y_A=\overline{d^{2n}A}\setminus d^{2n}A=d^{2n+1}A\setminus\overline{hd^{2n}A},\ n\geqq 1$$

$$Z_A=\overline{h^n dA}\setminus h^n dA,\ n\geqq 1$$

(h *is the coherence operator*: $hA=A\cap dA$)

that verify: *a*) $X_A\cap Y_A=\emptyset,\ X_A\cap Z_A=\emptyset,\ Y_A\cap Z_A=\emptyset$

b) $|X_A|=|Y_A|$

c) $X_A\cup Z_A=\overline{dA}\setminus dA$

REMARK. A space is a T_{D^2}-space if $X_A=\emptyset=Y_A$, for each subset A (see [2]). A space is a T_{HD}-space if $Z_A=\emptyset$, for each subset A (see [3]). $T_{HD}+T_{D^2}=T_D$.

3. DEFINTION. The sequence $A,\ dA,\ d^2A\cdots\cdots,\ d^{\lambda}A\cdots\cdots$ has *period of order* δ

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and *pre-period of order* α if $d^{\beta}A \neq d^{\gamma}A$, for $\alpha \leqq \beta < \gamma < \alpha+\delta$, and if $d^{\alpha}A = d^{\alpha+\delta}A$.

REMARK. Tucker [1] defines the periodicity without pre-period and only for sequences with finite ordinals. The generalizated periodicity defined above is not superfluous, as it is shown in the following exemple: Let X be the subset of the rationals numbers, with the usual order, $X = \left\{m + \frac{1}{n} \mid m, n \in Z,\ n \neq 0, \right\}$, with the topology given by the subbase of closed sets $\mathscr{C} = \left\{C_m, \left]-\infty,\ m+\frac{1}{n}\right] \mid m,\ n \in Z,\ n \neq 0,\ n \neq 2\right\}$, $C_m = \left\{m - \frac{1}{2},\ m\right\}$. For the subset $A = \left\{\frac{1}{3},\ \frac{1}{2}\right\}$ the sequence of its derived sets has pre-period ω^2 and period 1 : $d^{\omega^2}A = \bigcup_{m<\omega} C_{-m} = d^{\lambda} A$, $\lambda > \omega^2$.

With the preceding definition of periodicity, all sequences of derived sets of a subset become periodic.

4. PROPOSITION. *In a topological space X, the sequence of derived sets of a subset $A \subset X$ has period not greater than* 2.

PROOF. Let α be the pre-period of the sequence of derived sets of A. If $\alpha \geqq \omega$, then the period is 1, because $d^{\omega}A$ is closed (see [4]). If $\alpha < \omega$ and $d^{\alpha}A = d^{\lambda}A$ ($\lambda \geqq \omega$), then $d^{\alpha}A$ is closed and the period is 1. For $\alpha < \omega$ and finite period, if the period is even, then it is 2, and if the period is odd, then it is 1 (by proposition 2).

5. PROPOSITION. *A topological space X is T_{D^2} if and only if the sequence of derived sets of a subset $A \subset X$ has period not greater than* 1.

PROOF. Let α be the pre-period of the sequence of derived sets of A. Since d^2A is closed, if $\alpha \geqq 2$, the period is 1. If $\alpha = 1$ and $dA = d^nA$, for some $n \geqq 2$, then dA is closed and the period is 1. For $\alpha = 0$, *if* $A = d^nA$, for some $n \geqq 2$, then A is closed and the period is 1; if $A = dA$, the period is 1.

Inversely, if the space is not T_{D^2} there is a subset A such that d^2A is not closed. Then, the sequence of derived sets of $\overline{d^2A} \setminus d^2A$ has period 2 (and pre-period 0).

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