Kyungpook Math. J.
Volume 24, Number 1
June, 1984

ON CLOSED MODULES

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The concept of closed groups was first introduced by Kulikov [2] and later on a number of mathematicians like P. Hill, J. Irwin and E. Enochs etc. further worked on the same concept. The main purpose of this paper is to generalize the concept of closed group for the module M satisfying the following two conditions, known as S_2-module:

(i) Every finitely generated submodule of every homomorphic image of M is direct sum of uniserial modules.

(ii) Given any two uniserial modules U and V of a homomorphic image of M, for any submodule W of U, any nonzero homomorphism $f: W \longrightarrow V$ can be extended to a homomorphism $g: U \longrightarrow V$ provided the composition length $d(U/W) \leq d(V/f(W))$

All the modules considered here are unital and torsion over the associative ring with unity. For the basic concepts we refer the reader to [5, 6, 7].

Towards the definition of a closed module we define the following:

DEFINITION 1. Let M be an S_2-module without elements of infinite height. A sequence $x_1, x_2, \cdots x_n, \cdots$ of the elements of M is said to *converge to a limit* x if $x - x_k \in H_k(M)$ for every $k=1, 2, \cdots$

REMARK 1. Since M is free from the elements of infinite height this limit will be unique.

REMARK 2. Let $\{x_n\}$ and $\{x_n'\}$ be two sequences in M converging to x and x' respectively then $\{x_n \pm x_n'\}$ will converge to $x \pm x'$.

DEFINITION 2. Since every $x \in M$ can be uniquely written as a finite sum of uniform elements. We define h-exponent of an element $x \in M$ as follows:

$$h\text{-}\exp(x) = \max\{e(u_1),\ e(u_2), \cdots,\ e(u_n)\}$$

where $x = u_1 + u_2 + \cdots + u_n$ with u_i uniform.

DEFINITION 3. A sequence $\{x_n\}$ is said to be a *Cauchy-sequence* if $x_k - x_{k+1} \in H_k(M)$ for every k and h-exp (x_n) are bounded for every n.

REMARK 3. The sum and difference of two Cauchy sequences is also a Cauchy sequence.

DEFINITION 4. An S_2-module M without elements of infinite height is said to be *closed* if every Cauchy sequence in M has a limit in M.

REMARK 4. Intersection of two closed modules is a closed S_2-module.

As we know from [5] that if B is a basic submodule of M then $B=\oplus B_i$ where B_i's are the direct sum of uniserial modules of length i. The following theorem gives a characterization of closed S_2-modules.

THEOREM 1. *An S_2-module M is closed if and only if $M=\overline{B}$ where $\overline{B}=\Sigma B_i$, the complete direct sum of B_i's.*

PROOF. Let B be a basic sumodule of M and $B=\overset{\infty}{\underset{i=1}{\oplus}} B_i$. Suppose $M=\overline{B}$ and $\{x_n\}$ is a Cauchy sequence in $\overline{B}$ such that $x_n=(b_1^n, b_2^n, \cdots, b_k^n, \cdots)$ where $b_k^n \in B_k$ for every n. Trivially $x_k-x_{k+1}=(b_1^k-b_1^{k+1}, b_2^k-b_1^{k+1}, \cdots, b_n^k-b_n^{k+1}, \cdots)\in H_k(M)$. Since $b_i^k-b_i^{k+1}\in B_i$ and $x_k-x_{k+1}\in H_k(M)$, therefore $b_i^k-b_i^{k+1}=0$ for $i\leq k$ and $b_i^k-b_i^{k+1}\in H_k(M)$ for every $i>k$ and we infer that the first k components of x_k and x_{k+1} are identical. Considering the element $x=(b_1^1, b_2^2, \cdots b_k^k,)$ which is in $\overline{B}$, we have $x-x_k=(0, \cdots, 0, b_{k+1}^{k+1}-b_{k+1}^k, b_{k+2}^{k+2}-b_{k+2}^k, \cdots)$ such that $x-x_k\in H_k(M)$. Therefore x is the limit of $\{x_n\}$ and hence M is closed.

Conversely, suppose M is a closed S_2-module with a basic submodule B. Now appealing to [Prop 3.3.7] M is isomorphic to an h-pure submodule V of $\overline{B}$. Let $x=(b_1\ b_2, \cdots, b_k, \cdots)\in\overline{B}$, we define $x_n=b_1+\cdots+b_{k_n}$ where $k_n\geq n$ such that $b_k\in H_n(M)$ for every $k\geq k_n$. Now $h\text{-exp}(x_n)\leq k_n$ and $x_n-x_{n+1}\in H_n(M)\Longrightarrow\{x_n\}$ is a Cauchy sequence. If $x'=(b_1', \cdots, b_n', \cdots)$ is the limit of $\{x_n\}$ then $x'-x_n=(b_1'-b_1, b_2'-b_2, \cdots, b_{k_n}'-b_{k_n}, \cdots)\in H_n(M)$ therefore $b_1'-b_1=b_2'-b_2=\cdots=b_n'-b_n=0$ *i.e.* $b_n'=b_n$ for every n as $b_i'-b_i\in B_i$ and $x'=x$ is an element of M and $M=\overline{B}$.

COROLLARY 2. *Two closed S_2-modules are isomorphic if and only if their basic submodules are isomorphic.*

PROOF. Trivial.

REMARK 5. Let M be a S_2-module with a basic submodule B then $\overline{B}$ is defined to be the closure of M.

THEOREM 3. *Every direct summand of a closed S_2-module is closed and direct sum of a finite number of closed S_2-modules is closed.*

PROOF. Let M be a closed S_2-module such that $M=A\oplus A'$. If $\{x_n\}$ is a Cauchy sequence in A. Then its limit x is also an element of M where $x=x'+x''$, $x'\in A$, $x''\in A'$ and $x-x_n=x'-x_n+x''$ is an element of $H_n(M)$ for every n. Now $H_n(M)=H_n(A)\oplus H_n(A')$ i.e. $x'-x_n\in H_n(A)$ and $x''\in H_n(A')$ for every n and $x''=0$ implying $x'=x\in A$. For the second part it is sufficient to show that direct sum of two closed S_2-modules is also closed. Let $M=M'\oplus M''$ where M' and M'' are closed S_2-modules and B' and B'' are the basic submodules of M' and M'' respectively.

For completing the proof it is sufficient to show that for any basic submodule B of M, $M=\overline{B}$. Now appealing to theorem 1 $M'=\overline{B}'$, and $M''=\overline{B}''$. Since $\overline{B'\oplus B'}=\overline{B}'\oplus\overline{B'}$ we may take $M=\overline{B}'\oplus\overline{B'}$, but $B'\oplus B''$ is a basic submodule of M [Prop 3.4, 7] and by [th. 2, 5] B is isomorphic to $B'\oplus B''$. Therefore we can say that $M=\overline{B}$ and M is closed.

Now we are able to generalize the following results proved by P. Hill [3], J. Irwin [4] and Enochs [1].

NOTATION. $\sum B_i$ denotes the complete direct sum of B_i's and $\oplus B_i$ is the discrete direct sum of B_i's and $\oplus\sum_{i=1}^{\infty} B_i$ denotes the unspecified sum of B_i's.

THEOREM 4. *Each subsocle of a closed module supports an h-pure submodule.*

PROOF. Let B be a basic submodule of M such that $M=\overline{B}$ and $B=\bigoplus_{i=1}^{\infty} B_i$. Now soc $(M)=\sum_{i=1}^{\infty}(\text{soc }(B_i))$. Suppose A is a subsocle of M, then A will be decomposable and $A=\oplus\sum\text{soc }(B_{ij})$ for some i, j, then choosing $N=\oplus\sum B_{ij}$ for the same indices we get N to be an h-pure submodule of M and hence the result follows.

THEOREM 5. *An S_2-module without elements of infinite height is closed if and only if its socle is closed.*

PROOF. Suppose soc (M) is closed i.e. soc (M) is the complete direct sum of its summands. We will prove this result by induction. Let M_k be the submodule of M generated by the uniform elements of exponent atmost k then $M=\bigcup_{k=1}^{\infty} M_k$ and M_k is a basic submodule of itself i.e. M_k is a direct sum of uniserial modules. Suppose M_i is closed for every $i<k$. Clearly $M_1=\text{soc}(M)$ is closed. Consider M_{k-1} which is closed, hence it is a complete direct sum of its summands and we write $M_{k-1}=\overline{M}_{k-1}$. But $\frac{M_k}{M_1}$ is closed implying $\frac{M_k}{M_1}=\overline{\left(\frac{M_k}{M_1}\right)}$. Now it is easy to check that $\frac{\overline{M}_k}{M_1}=\overline{\left(\frac{M_k}{M_1}\right)}$ which implies $M_k=\overline{M}_k$ and M_k is closed. Therefore the result follows.

Conversely, suppose M is closed then $M=\overline{B}=\sum B_i$ where B_i is the direct sum of the uniserial modules of length i. Now *soc* (M) is also the complete direct sum of its summands and hence it is closed.

THEOREM 6. *Let M be a closed S_2-module such that S is a subsocle of M and K is a submodule of M such that soc$(K)\subset S$. There exists an h-pure submodule T of M that $K\subseteq T$ and* $\text{soc}(T)=S$

PROOF. Appealing to theorem 4 S supports an h-pure submodule. Consider the family $\mathscr{F}$ of the h-pure submodules N_i of M such that $K\subseteq N_i\subseteq N_{i+1}$ and $\text{soc}(N_i)\subseteq S$, $i=1,2,\cdots$ Since M is free from the elements of infinite height this family will be finite. Suppose T is the maximal element of $\mathscr{F}$ then $\text{soc}(T)\subseteq S$. Let $x\in S$ and $x\notin T$. Consider $A=xR\oplus T$ then $\frac{A}{T}\cong xR$ and hence $d\left(\frac{A}{T}\right)=1$. If $\Theta: M\longrightarrow\frac{M}{T}$ is the natural epimorphism then the inverse image of $\frac{A}{T}$ will contain T and x contradicting the maximality of T implying that *soc* $(T)=S$.

Every direct summand of an S_2-module is h-pure but the converse is not true in general. By defining h-reduced modules, we are giving a solution of this problem.

DEFINITION 6. An S_2-module is said to be *h-reduced* if it does not contain an h-divisible sumbodule [6].

THEOREM 7. *An h-reduced S_2-module M is a direct summand of every S_2-module N in which it is h-pure if and only if M is closed.*

PROOF. Suppose M is an S_2-module with the stated property. Let $x_1\in M$ such that $H(x_1)=\infty$ i.e. there exist elements x_2, x_3, ……, x_n, …… in M such that

$d\left(\frac{x_2R}{x_1R}\right)=1$, $d\left(\frac{x_3R}{x_2R}\right)=1, \cdots$, $d\left(\frac{x_nR}{x_{n-1}R}\right)=1$ and so on. Now the submodule generated by x_1, $x_2, \cdots$ will be h-divisible which is a contradiction, implying that M is free from the elements of infinite height.

If B is a basic submodule of M then by [Prop. 3.3, 7] M can be embedded in $\overline{B}$ as h-pure submodule. Suppose M is not closed then $M \neq \overline{B}$, but $B \subseteq M$ and M is not a direct summand of $\overline{B}$, which is a contradiction, hence M is closed.

Conversely, suppose M is closed and $M \subset N$ such that M is h-pure in N. Let B' be a basic submodule of M and $B' = \oplus b_{ij}R$ such that $(b_{ij})_{i,j=1,2,\cdots}$ is a h-pure independent set in B', which can be extended to a maximal h-pure independent subset of N $(b_{ij}', b_{ij}'')_{i,j=1,2,\cdots}$. If $B'' = \oplus b_{ij}''R$ and $B = B' \oplus B''$ then N is the homomorphic image of $\overline{B} = \overline{B}' \oplus \overline{B}''$ i.e. $\eta(\overline{B}) = N$ for some η then η is a projection of $\overline{B}'$ and $M = \overline{B}'$ is a direct summand of N.

Using the facts, stated above we can easily infer the following results:

THEOREM 8. *Let M be an S_2-module such that* $M = \bigoplus_{i=1}^{\infty} M_i$ where each M_i *is closed. Suppose N is a closed summand of M_i. Then* $N = \bigoplus_{i=1}^{\infty} N_i$ *where N_i's are closed.*

PROOF. we define $S_n = \text{soc}(N \cap (\sum_{i=1}^{n} M_i))$. Since $\sum_{i=1}^{n} M_i$ is closed S_n will support an h-pure submodule T_n which is contained in $\sum_{i=1}^{n} M_i$. Now $N \cap (\sum_{i=1}^{n} M_i)$ is closed and therefore soc $(N \cap (\sum_{i=1}^{n} M_i))$ is also closed. Again by the same argument T_n is also closed as soc $(T_n) = S_n$ which is closed such that $T_1 \subseteq T_2 \subseteq T_3 \subseteq \cdots\cdots$

Consider the natural projection $\pi : M \longrightarrow N$, then $\pi(T_n) \cong T_n$ is closed and h-pure in N and therefore a summand of N. Now we have a chain of closed summands of N namely $\pi(T_1) \subseteq \pi(T_2) \subseteq \cdots$ where $\cup \pi(T_i) = N$ and we define N_i as follows:

$$\pi(T_{i-1}) \oplus N_i = \pi(T_i) \text{ if } i > 1 \text{ and } N_1 = \pi(T_1)$$

THEOREM 9. *Let $M' \oplus M''$ be an S_2-module and N is a h-pure closed submodule of M such that* soc $(N) \subseteq M'$. *Then M' admits a direct decomposition $M' = N' \oplus N''$ such that* $\text{soc}(N') \subseteq \text{soc}(N)$ *and* $N' \cong N$.

PROOF. Let π be the projection determined by the direct decomposition onto M'. Since N is h-pure closed submodule of M, $\pi(N)$ is also h-pure and closed and by theorem 7, $\pi(N)$ is a direct summand of M. Suppose $\pi(N)=N'$ then $N'=N$ and $M'=N'\oplus N''$ where N'' is the supplement of N' in M'.

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