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ON CHARACTERS OF η-RELATED TENSORS IN COSYMPLECTIC AND SASAKIAN MANIFOLDS (3)*

By Sang-Seup Eum

§1. Introduction.

Let M be a $(2n+1)$-dimensional differentiable manifold covered by a system of coordinate neighborhoods $\{U, x^h\}$, where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$ and let M admits an almost contact metric structure, that is, a set $(\phi_i^{\ h}, \xi^h, \eta_i, g_{ji})$ of a tensor field $\phi_i^{\ h}$ of type (1,1), a vector field ξ^h, a 1-form η_i and a positive definite Riemannian metric g_{ji} satistying

$$\phi_j^{\ i}\phi_i^{\ h}=-\gamma_j^{\ h},\qquad \phi_t^{\ i}\xi^t=0,\qquad \eta_t\phi_j^{\ t}=0,\qquad \eta_t\xi^t=1,$$

$$g_{st}\phi_j^{\ s}\phi_i^{\ t}=\gamma_{ji},\qquad \eta_i=g_{it}\xi^t, \tag{1.1}$$

where

$$\gamma_{ji}=g_{ji}-\eta_j\eta_i,\qquad \gamma_j^{\ h}=g^{ht}\gamma_{jt}. \tag{1.2}$$

A manifold with such a structure is called an *almost contact metric manifold*. By virtue of the last equation of (1.1), we shall write η^h instead of ξ^h in the sequel.

In an almost contact metric manifold M, we define an *η-holomorphically projective vector field v^h* by the condition

$$\begin{aligned} L_v\{{}_k{}^h{}_j\} &= \nabla_k\nabla_j v^h + v^t K_{tkj}^{\ \ \ \ h} \\ &= \gamma_k^{\ h}p_j+\gamma_j^{\ h}p_k-p_t(\phi_k^{\ t}\phi_j^{\ h}+\phi_j^{\ t}\phi_k^{\ h}), \end{aligned} \tag{1.3}$$

for a certain covector field p_i, called the associated covector field of v^h, where $\{{}_k{}^h{}_j\}$, $K_{kjt}^{\ \ \ \ h}$, ∇_k and L_v are respectively the Christoffel symbols formed with g_{ji}, the curvature tensor of M, the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$ and the operator of Lie derivation with respect to v^h.

In the present paper, we call an η-holomorphically projective vector field

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briefly an *η-HP vector field.*

The purpose of the present paper is to investigate the properties of η-HP vector fields in cosymplectic and Sasakian manifolds.

§2. Cosymplectic manifolds.

A normal almost contact metric manifold is said to be *cosymplectic* if the 2-form $\phi_{ji}=\phi_j{}^t g_{ti}$ and 1-form η_i are both closed.

It is well known that the cosymplectic structure is characterized by

$$\nabla_k \phi_j{}^h=0, \qquad \nabla_k \eta_i=0. \tag{2.1}$$

In a $(2n+1)$-dimensional cosymplectic manifold M, we easily see that

$$K_{kjit}\eta^t=0, \qquad K_{jt}\eta^t=0 \tag{2.2}$$

by virtue of the Ricci identity with respect to η^h, where K_{ji} is the Ricci tensor of M. Moreover, using the Ricci identity with respect to $\phi_i{}^h$, the following equations are satisfied.

$$K_{kjts}\phi_i{}^t\phi_h{}^s=K_{kjih}, \tag{2.3}$$

$$K_{kjts}\phi^{ts}=2K_{kt}\phi_j{}^t, \tag{2.4}$$

$$K_{jt}\phi_i{}^t+K_{it}\phi_j{}^t=0, \qquad K_{ts}\phi_j{}^t\phi_i{}^s=K_{ji}. \tag{2.5}$$

Now we define tensor fields G_{ji} and $Z_{kji}{}^h$ on M by respectively

$$G_{ji}=K_{ji}-\frac{K}{2n}\gamma_{ji}, \tag{2.6}$$

$$Z_{kji}{}^h=K_{kji}{}^h-\frac{K}{4n(n+1)}(\gamma_k{}^h\gamma_{ji}-\gamma_j{}^h\gamma_{ki}+\phi_k{}^h\phi_{ji}-\phi_j{}^h\phi_{ki}-2\phi_{kj}\phi_i{}^h), \tag{2.7}$$

where K is the scalar curvature of M.

The following equations are also satisfied.

$$G_t{}^t=g^{ht}G_{th}=0, \qquad Z_{tji}{}^t=G_{ji}, \tag{2.8}$$

$$Z_{kjih}=-Z_{jkih}, \qquad Z_{kjth}=Z_{ihkj}, \qquad Z_{kjit}\eta^t=0, \tag{2.9}$$

where $Z_{kjih}=Z_{kji}{}^t g_{th}$,

and

$$G_{jt}\eta^t=0, \qquad G_{jt}\phi_k{}^t+G_{kt}\phi_j{}^t=0, \quad G_{ts}\phi_j{}^t\phi_i{}^s=G_{ji}. \tag{2.10}$$

Taking account of the fact that all of the Lie derivatives of g_{ji}, $\left\{ {h \atop k\ j} \right\}$, $K_{kji}{}^h$, and K_{ji} with respect to η^h are vanish, we obtain respectively

$$\eta^t\nabla_t K_{kji}{}^h=0,\ \eta^t\nabla_t K_{ji}=0,\ \eta^t\nabla_t K=0,$$

and from which

$$\eta^t\nabla_t G_{ji}=0. \tag{2.11}$$

The following equations are also berified.

$$Z_{kjih}\gamma^{ji}=G_{kh},\qquad Z_{kjih}\gamma^{ki}=-G_{jh}, \tag{2.12}$$

$$Z_{kji}{}^t\phi_t{}^h=Z_{kjt}{}^h\phi_i{}^t,\qquad Z_{kti}{}^h\phi_j{}^t=Z_{jti}{}^h\phi_k{}^t. \tag{2.13}$$

$$Z_{kjih}\phi^{ki}=K_{jt}\phi_h{}^t+\frac{K}{2n}\phi_{jh}, \tag{2.14}$$

$$Z_{kjih}\phi^{kj}=2K_{it}\phi_h{}^t+\frac{K}{n}\phi_{ih}, \tag{2.15}$$

and

$$Z_{kjih}\phi^{ki}\phi_s{}^j=G_{hs},\qquad Z_{kjih}\phi^{kj}\phi_s{}^i=2G_{hs}. \tag{2.16}$$

If the scalar curvature K of M is a constant, then we obtain the following equations.

$$\nabla^t K_{tj}=\frac{1}{2}\nabla_j K=0,\qquad \nabla_t K_{kji}{}^t=\nabla_k K_{ji}-\nabla_j K_{ki}, \tag{2.17}$$

$$\nabla^t G_{tj}=0, \tag{2.18}$$

and

$$\nabla_t Z_{kji}{}^t=\nabla_k G_{ji}-\nabla_j G_{ki} \tag{2.19}$$

by virtue of the second Bianchi identity for $Z_{kji}{}^h$.

§3. η-*HP* vector fields in a cosymplectic manifold.

In a previous paper [1], we proved that following

THEOREM A. *If M is a cosymplectic manifold of constant ϕ-holomorphic sectional curvature, then $Z_{kji}{}^h$ in M defined by (2.7) vanishes.*

We consider a system of differential equations

$$\nabla_k\nabla_j p_h=-\frac{K}{4n(n+1)}\{2\gamma_{jh}p_k+\gamma_{kh}p_j+\gamma_{kj}p_h -p_t(\phi_j{}^t\phi_{kh}+\phi_h{}^t\phi_{kj})\} \tag{3.1}$$

for a certain covector field p_h in a cosymplectic manifold M of constant ϕ-holomorphic sectional curvature.

If p^h belongs to the distribution orthogonal to η^h ([2]), that is, $p_t\eta^t=0$ in M, then, it is easily seen that the integrability condition of (3.1) is satisfied by the help of theorem A. In this case, taking account of (3.1) and the fact that the right hand member of (2.7) vanishes, we obtain

$$(3.2)\qquad L_p\left\{{h \atop k\,j}\right\}=\nabla_k\nabla_j p^h+p^t K_{tkj}{}^h$$
$$=-\frac{K}{2n(n+1)}\{\gamma_k{}^h p_j+\gamma_j{}^h p_k-p_t(\phi_k{}^t\phi_j{}^h+\phi_j{}^t\phi_k{}^h)\},$$

where L_p denotes the Lie derivation with respect to p^h.

In a previous paper [2], we proved that the distribution orthogonal to η^h is integrable in a cosymplectic manifold M. Therefore, there exists locally a vector field p^h in M such that $p_t\eta^t=0$. Thus we have the following.

THEOREM 3.1. *In a cosymplectic manifold M of constant ϕ-holomorphic sectional curvature, there exists locally an η-HP vector field p^h whose associated covector is proportional to itself.*

§4. η-*HP* vector fields in compact cosymplectic manifolds.

Here-after, we assume that M is a $(2n+1)$-dimensional compact cosymplectic manifold.

We notice that every almost contact manifold is orientable. (cf. [4], 1—13, Theorem 1.5.)

Transvecting (1.3) with η^h and taking account of (2.2), we obtain $\nabla_k\nabla_j(\eta_t v^t)=0$, from which

$$(4.1)\qquad \eta_t v^t=\text{constant}$$

by the help of the theorem (1.4) of p. 24 in [6].

Thus, for an η-*HP* vector field in M, we obtain

$$(4.2)\qquad \eta_t\nabla_k v^t=0,\ \text{i.e.},\ L_v\eta=0.$$

Transvecting (1.3) with g^{kj}, we obtain

$$(4.3)\qquad \nabla^t\nabla_t v^i+K_t{}^i v^t=0.$$

For a vector field v^h satisfying (4.1) in M, we obtain ([3])

$$\frac{1}{2}(\nabla_j v_i-\phi_j{}^t\phi_i{}^s\nabla_t v_s)(\nabla^j v^i-\phi^{jk}\phi^{il}\nabla_k v_l)$$

$$=(\nabla^j v^i)(\nabla_i v_j)-\phi^{jt}\phi^{is}(\nabla_j v_i)(\nabla_t v_s)-\frac{1}{2}(\eta^t\nabla_t v^j)(\eta^s\nabla_s v_j),$$

and from which

$$\nabla_j[(\nabla^j v^i)v_i-\phi^{jt}\phi^{is}(\nabla_t v_s)v_i]$$
$$=\frac{1}{2}(\nabla_j v_i-\phi_j^{\ t}\phi_i^{\ s}\nabla_t v_s)(\nabla^j v^i-\phi^{jt}\phi^{is}\nabla_t v_s)$$

by the help of (4.3).

Integrating this over M and taking account of the theorem of Green, we obtain

$$\nabla_j v_i-\phi_j^{\ t}\phi_i^{\ s}\nabla_t v_s=0$$

or equivalently

$$L_v\phi_j^{\ h}=0. \tag{4.4}$$

Operating the Lie derivation with respect to v^h to the first equation of (1.1) and taking account of (4.2) and (4.4), we obtain

$$L_v\eta^h=0. \tag{4.5}$$

Therefore v^h is a contravariant C^*-analytic vector field in M. (cf. definition in [3]).

Substituting (1.3), (2.1) and (4.4) into the identity

$$L_v(\nabla_j\phi_i^{\ h})-\nabla_j(L_v\phi_i^{\ h})=(L_v\{{}^{\ h}_{j\ t}\})\phi_i^{\ t}-(L_v\{{}^{\ t}_{j\ i}\})\phi_t^{\ h},$$

we obtain

$$(p_t\eta_i^{\ t}-p_i)\phi_j^{\ h}=0.$$

Transvecting this equation with $\eta^i\phi_h^{\ j}$ we obtain

$$\eta_t p^t=0. \tag{4.6}$$

Contracting on h and j in (1.3) and taking account of (4.6), we see that

$$p_j=\nabla_j p, \tag{4.7}$$

where $p=\frac{1}{2(n+1)}\nabla_t v^t$.

Thus we have the following

THEOREM 4.1. *If a compact cosymplectic manifold M admits an η-HP vector field v^h, then v^h is a contravariant C^*-analytic vector, the associate vector p^h of v^h is orthogonal to η^h and p_i is a gradient vector.*

For an η-HP vector field v^h, the following equations are satisfied.

(4.8) $$\nabla_k L_v g_{jh} = 2\gamma_{jh} p_k + \gamma_{kh} p_j + \gamma_{kj} p_h - p_t(\phi_j{}^t \phi_{kh} + \phi_h{}^t \phi_{kj}).$$

(4.9) $$\nabla_k L_v g^{jh} = -2\gamma^{jh} p_k - \gamma_k{}^j p^h - \gamma_k{}^h p^j - p^t(\phi_t{}^j \phi_k{}^h + \phi_t{}^h \phi_k{}^j),$$

where $p^t = g^{th} p_h$.

Substituting (1.3) into the well known formula

$$L_v K_{kji}{}^h = \nabla_k L_v \{{}^h_{j\,i}\} - \nabla_j L_v \{{}^h_{k\,i}\},$$

we find

(4.10) $$L_v K_{kji}{}^h = \gamma_j{}^h \nabla_k p_i - \gamma_k{}^h \nabla_j p_i - (\nabla_k p_t)(\phi_j{}^t \phi_i{}^h + \phi_i{}^t \phi_j{}^h) + (\nabla_j p_t)(\phi_k{}^t \phi_i{}^h + \phi_i{}^t \phi_k{}^h),$$

and from which

(4.11) $$L_v K_{ji} = -2n \nabla_j p_i - 2(\nabla_t p_s)\phi_j{}^t \phi_i{}^s.$$

Transvecting (4.11) with $\phi_t{}^j \phi_s{}^i$ and taking account of the second equation of (2.5), we obtain

(4.12) $$L_v K_{ts} = -2n(\nabla_j p_i)\phi_t{}^j \phi_s{}^i - 2\nabla_t p_s.$$

Comparing (4.11) with (4.12), we obtain

(4.13) $$\nabla_j p_i = (\nabla_t p_s)\phi_j{}^t \phi_i{}^s,$$

or equivalently

(4.14) $$L_p \phi_j{}^k = 0.$$

Substituting (4.13) into (4.11), we obtain

(4.15) $$L_v K_{ji} = -2(n+1)\nabla_j p_i.$$

Operating ∇_k to (4.13) and transvecting it with g^{kj}, we obtain

(4.16) $$\nabla^t \nabla_t p_i = \phi_i{}^t \phi^{ks} \nabla_k \nabla_s p_t = -\frac{1}{2}\phi_i{}^t \phi^{ks} K_{kst}{}^h p_h$$

by virtue of the Ricci identity

Substituting (2.4) into (4.16), we obtain

(4.17) $$\nabla^t \nabla_t p^h + K_t{}^h p^t = 0.$$

Taking account of (2.6) and (4.15), we obtain

(4.18) $$L_v G_{ji} = -(\nabla_j w_i + \nabla_i w_j),$$

where we have put

(4.19) $$w^h=(n+1)p^h+\frac{K}{2n}v^h.$$

Taking account of (2.7) and (4.10), we obtain

(4.20) $$L_vZ_{kji}{}^h=\frac{1}{2(n+1)}[\gamma_k{}^hL_vG_{ji}-\gamma_j{}^hL_vG_{ki}+(L_vG_{kt})(\phi_t{}^j\phi_i{}^h+\phi_i{}^t\phi_j{}^h)-(L_vG_{jt})(\phi_k{}^t\phi_i{}^h+\phi_i{}^t\phi_k{}^h)].$$

§5. Decomposition of an η-*HP* vector field in *M*.

In this section, we prove the following

THEOREM 5.1. *If a* $(2n+1)$*-dimensional compact cosymplectic manifold with constant scalar curvature* K *admits an* η-*HP vector field* v^h, *then the following propositions are satisfied.*

(1) $K>0$.

(2) v^h *is decomposed uniquely in the form*

$$v^h=\frac{2n}{K}\{w^h-(n+1)p^h\},$$

where w^h *is a Killing vector field and* p^h *is the associated vector of* v^h.

(3) p^h *is also an* η-*HP vector and whose associated vector is proportional to itself.*

LEMMA 1. *If* w^h *defined by* (4.19) *is a Killing vector field, then theorem* (5.1) *is satisfied.*

Proof of lemma 1.

(1) Differentiating covariantly (4.19) and taking account of (4.2), we obtain

(5.1) $$2(n+1)\nabla_jp_i=\frac{K}{2n}(\nabla_jv_i+\nabla_iv_j)=0.$$

Operating ∇_k to (5.1) and substituting (1.3) into it, we obtain

(5.2) $$2(n+1)\nabla_k\nabla_jp_i=\frac{K}{2n}[2\gamma_{ji}p_k+\gamma_{ki}p_j+\gamma_{jk}p_i-p_l(\phi_j{}^t\phi_{ki}+\phi_i{}^t\phi_{kj})].$$

Transvecting (5.2) with g^{ji}, we obtain

$$\nabla_k\nabla_tp^t=-\frac{K}{n}p_k,$$

from which, we see that $-\frac{K}{n}<0$. (cf. theorem 1.8 of p.26 in [6]). Thus we see that $K>0$.

(2) From (4.19), we see that

$$v^h=\frac{2n}{K}\{w^h-(n+1)p^h\}.$$

For proof of the uniqueness of above decomposition, if we put

$$\frac{2n}{K}\{w^h-(n+1)p^h\}=\frac{2n}{K}\{'w^h-(n+1)'p^h\},$$

then $'p^h-p^h$ also a Killing vector. On the other hand $'p_i-p_i$ is a gradient vector by virtue of (4.7). Thus we have $\nabla_j('p_i-p_i)=0$ and from which $\nabla_j\nabla_i('p-p)=0$, where $'p_i=\nabla_i'p$. Then using the theorem 1.4 of p. 24 in [6], we see that $'p-p$ $=$constant and from which $'p_i=p_i$. Therefore, the uniqueness above stated is proved.

(3) Differentiating (4.19) covariantly, we have

$$\nabla_k\nabla_j w^h=(n+1)\nabla_k\nabla_j p^h+\frac{K}{2n}\nabla_k\nabla_j v^h, \tag{5.3}$$

from which,

$$L_w\{^{\,h}_{k\,j}\}=(n+1)L_p\{^{\,h}_{k\,j}\}+\frac{K}{2n}L_v\{^{\,h}_{k\,j}\}, \tag{5.4}$$

where L_w indicates the Lie derivation with respect to w^h.

Since w^h is a Killing vector, the left hand member of (5.4) vanishes. Substituting (1.3) into (5.4), we obtain.

$$L_p\{^{\,h}_{k\,j}\}=-\frac{K}{2n(n+1)}[\gamma_k{}^h p_j+\gamma_j{}^h p_k-p_t(\phi_k{}^t\phi_j{}^h+\phi_j{}^t\phi_k{}^h)], \tag{5.5}$$

and we are done.

Next, we are going to the proof of the fact that w^h defined by (4.19) is a Killing vector field.

For this purpose, we use for briefness the following notations.

$$\mu=(\nabla_t w^t)^2, \tag{5.6}$$

$$\nu=(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j). \tag{5.7}$$

From (4.19), we obtain

$$\nabla^t\nabla_t w^h+K_t{}^h w^t=(n+1)(\nabla^t\nabla_t p^h+K_t{}^h p^t)+\frac{K}{2n}(\nabla^t\nabla_t v^h+K_t{}^h v^t).$$

Substituting (4.3) and (4.17) into this equation, we obtain

$$\nabla^t\nabla_t w^h+K_t{}^h w^t=0, \tag{5.8}$$

from which by a well known integral, we obtain the following

LEMMA 2. $\int_M \mu dV = \frac{1}{2}\int_M \nu dV$,

where dV is the volume element of M.

Taking account of the Ricci identity for p^t and (4.17), we obtain

$$\nabla_i \nabla_t p^t - \nabla_t \nabla_i p^t = -K_{it} p^t.$$

Using this fact, (2.6), (4.6) and lemma 2, we obtain the following

LEMMA 3. $\int_M G_{ji} p^j w^i dV = \frac{1}{4(n+1)}\int_M \nu dV$.

By the same way as §3 in [2] (cf. (3.6) of [2]), we obtain the following

LEMMA 4. $\int_M (\nabla^j L_v G_{ji}) w^i dV = \frac{1}{2}\int_M \nu dV$.

Using (1.3), (2.10) and (2.11), we obtain

$$g^{kj} L_v \nabla_k G_{ji} = \nabla^j L_v G_{ji} - 2G_{ji} p^j.$$

Taking account of this equation and lemma 3 and lemma 4, we obtain the following

LEMMA 5. $\int_M g^{kj}(L_v \nabla_k G_{ji}) w^i dV = \frac{n}{2(n+1)}\int_M \nu dV$.

Taking account of (2.6), (2.8), (4.9) and (4.18), we obtain

$$G_{ji} \nabla_k L_v g^{ji} = -4G_k{}^t p_t,$$

and

$$G_{ji} L_v g^{ji} = 2\nabla_t w^t.$$

Therefore using lemma 2 and lemma 3, we obtain the following

LEMMA 6. $\int_M (\nabla^h G_{ji})(L_v g^{ji}) w_h dV = -\frac{n}{n+1}\int_M \nu dV$.

Taking account (2.18) and lemma 5, we obtain the following

LEMMA 7. $\int_M (\nabla_i G_{hj})(L_v g^{ji}) w^h dV = -\frac{n}{2(n+1)}\int_M \nu dV$.

Operating ∇^k to (4.20) and taking account of (2.8), (2.10), (2.11), (2.12), (4.9) and lemmas 3, 4 and 6, we have the following

LEMMA 8. $\int_M (\nabla^k L_v Z_{kji}{}^h) g^{ji} w_h dV = \frac{4n+3}{4(n+1)^2} \int_M \nu dV.$

Similarly, using (2.9), (2.12), (2.14), (2.15), (2.16), (4.9) and the lemmas 3, 4, 5 and 6, we can obtain the following

LEMMA 9. $\int_M (\nabla^k L_v Z_{kjih}) g^{ji} w^h dV = \frac{2n+3}{2(n+1)} \int_M \nu dV.$

Now we prove the theorem 5.1.

Taking account of the last equation of (2.9), (2.12) and (2.18), we obtain

$$(\nabla^k Z_{kji}{}^h) g^{ji} = \nabla^k G_k{}^h = 0,$$

and from which,

$$(\nabla^k L_v Z_{kjih}) g^{ji} w^h = (\nabla^k L_v Z_{kji}{}^t) g^{ji} w_t + G_k{}^t (\nabla^k L_v g_{th}) w^h.$$

Substituting (4.8) into this and using lemmas 3 and 8, we obtain

$$\int_M (\nabla^k L_v Z_{kjih}) g^{ji} w^h dV = \frac{3}{2(n+1)} \int_M \nu dV.$$

Comparing this equation with lemma 9, we obtain

$$\int_M \nu dV = 0,$$

that is

$$\nabla_j w_i + \nabla_i w_j = 0. \tag{5.9}$$

Therefore, the theorem 5.1 follows from lemma 1.

Taking account of (4.18) and (5.9), we obtain

$$L_v G_{ji} = 0 \tag{5.10}$$

and from which

$$L_v Z_{kji}{}^h = 0. \tag{5.11}$$

Taking account of (2.6), (4.2) and (5.10), we obtain

$$L_v K_{ji} = \frac{K}{2n} L_v g_{ji}. \tag{5.12}$$

Substituting (5.12) into (4.15), we obtain

$$\frac{K}{2n} L_v g_{ji} = -2(n+1) \nabla_j p_i. \tag{5.13}$$

Differentiating covariantly (5.13), considering the assumption K=const. and taking account of (4.8), we obtain

$$\nabla_k \nabla_j p_i = -\frac{K}{4n(n+1)} \{2\gamma_{ji} p_k + \gamma_{ki} p_j + \gamma_{kj} p_i \tag{5.14}$$

$$-p_t(\phi_j{}^t\phi_{ki}+\phi_i{}^t\phi_{kj})\}.$$

Substituting (5.14) into the Ricci identity, we obtain

$$K_{kji}{}^t p_t=\frac{K}{4n(n+1)}(\gamma_{ji}\gamma_k{}^t-\gamma_{ki}\gamma_j{}^t+\phi_{ji}\phi_k{}^t-\phi_{ki}\phi_j{}^t-2\phi_{kj}\phi_i{}^t)p_t, \tag{5.15}$$

and from which

$$Z_{kji}{}^t p_t=0. \tag{5.16}$$

Thus we have the following

THEOREM 5.2. *Let M be a compact cosymplectic manifold with constant scalar curvature.*

If the Lie algebra of all η-HP vectors is transitive, then M is of constant φ-holomorphic sectional curvature.

Transvecting the first equation of (2.13) with p_h and taking account of (5.16), we obtain

$$Z_{kji}{}^t\phi_t{}^h p_h=0. \tag{5.17}$$

Substituting (5.11) into the identity

$$L_v\nabla_l Z_{kji}{}^h-\nabla_l L_v Z_{kji}{}^h$$
$$=Z_{kji}{}^t L_v\{{}^{\,h}_{l\ t}\}-Z_{tji}{}^h L_v\{{}^{\,t}_{l\ k}\}-Z_{kti}{}^h L_v\{{}^{\,t}_{l\ j}\}-Z_{kjt}{}^h L_v\{{}^{\,t}_{l\ i}\},$$

taking account of (1.3), (2.7), (2.14), (5.16), (5.17) and transvecting it with p^k, we obtain

$$(L_v\nabla_l K_{kji}{}^h)p^k=-Z_{lji}{}^h(p_k p^k). \tag{5.18}$$

Contracting by $h=j$ in (5.18), we obtain

$$(L_v\nabla_l K_{ki})p^k=-G_{li}(p_k p^k). \tag{5.19}$$

Taking account of (5.18) and (5.19), we obtain the following

THEOREM 5.3. *Let M be a compact cosymplectic manifold with constant scalar curvature, and let M admits an η-HP vector field.*

(1) *If M is symmetric manifold, then M is of constant φ-holomorphic sectional curvature.*

(2) *If M is Ricci parallel, then M is an η-Einstein manifold.*

§6. An η-HP vector field in a Sasakian manifold.

If a set $(\phi_i^{\ h},\ \eta^h,\ g_{ji})$ of a tensor field $\phi_i^{\ h}$ of type (1.1), a vector field η^h and a Riemannian metric g_{ji} satisfies (1.1), (1.2) and additionally

$$\phi_{ji}=\frac{1}{2}(\partial_j\eta_i-\partial_i\eta_j),$$

then such a set is called a contact structure. A manifold with normal contact structure is called a Sasakian manifold.

It is well known that in a Sasakian manifold, the following equations are satisfied.

$$\text{(6.1)}\qquad \nabla_i\eta^h=\phi_i^{\ h},\ \nabla_j\phi_i^{\ h}=-g_{ji}\eta^h+\delta_j^{\ h}\eta_i,$$

$$\text{(6.2)}\qquad \eta_t K_{kji}^{\ \ \ t}=\eta_k g_{ji}-\eta_j g_{ki},$$

and

$$\text{(6.3)}\qquad K_{jt}\eta^t=2n\eta_j.$$

In the present section, we investigate an η-HP vector field defined by (1.3), that is

$$\text{(6.4)}\qquad \begin{aligned} L_v\{{}_k{}^h{}_j\} &=\nabla_k\nabla_j v^h+v^t K_{tkj}^{\ \ \ h}\\ &=p_k\gamma_j^{\ h}+p_j\gamma_k^{\ h}-p_t(\phi_k^{\ t}\phi_j^{\ h}+\phi_j^{\ t}\phi_k^{\ h}),\end{aligned}$$

in a Sasakian manifold.

Differentiating (6.4) covariantly, we obtain

$$\text{(6.5)}\qquad \begin{aligned}\nabla_k L_v\{{}_j{}^h{}_i\} = &-(\phi_{kj}\eta^h+\phi_k^{\ h}\eta_j)p_i-(\phi_{ki}\eta^h+\phi_k^{\ h}\eta_i)p_j\\ &-(\nabla_k p_t)(\phi_j^{\ t}\phi_i^{\ h}+\phi_i^{\ t}\phi_j^{\ h})+(p_t\eta^t)(g_{kj}\phi_i^{\ h}\\ &+g_{ki}\phi_j^{\ h})-p_k(\eta_j\phi_i^{\ h}+\eta_i\phi_j^{\ h})\\ &+p_t\phi_j^{\ t}(\eta^h g_{ki}-\eta_i\delta_k^{\ h})+p_t\phi_i^{\ t}(\eta^h g_{kj}-\eta_j\delta_k^{\ h}).\end{aligned}$$

Substituting (6.5) into the identity

$$L_v K_{kji}^{\ \ \ h}=\nabla_k L_v\{{}_j{}^h{}_i\}-\nabla_j L_v\{{}_k{}^h{}_i\},$$

we obtain

$$\text{(6.6)}\qquad \begin{aligned} L_v K_{kji}^{\ \ \ h}= &-(2\phi_{kj}\eta^h+\phi_k^{\ h}\eta_j-\phi_j^{\ h}\eta_k)\eta_i\\ &-(\phi_{ki}p_j-\phi_{ji}p_k)\eta^h-(\nabla_k p_t)(\phi_j^{\ t}\phi_i^{\ h}+\phi_i^{\ t}\phi_j^{\ h})\\ &+(\nabla_j p_t)(\phi_k^{\ t}\phi_i^{\ h}+\phi_i^{\ t}\phi_k^{\ h})+p_t\eta^t(g_{ik}\phi_j^{\ h}\end{aligned}$$

$$-g_{ji}\phi_k^{\ h})+p_t\phi_j^{\ t}(\eta^h g_{ki}-\eta_i\delta_k^{\ h})$$
$$-p_t\phi_k^{\ t}(\eta^h g_{ji}-\eta_i\delta_j^{\ h})-(\eta_j p_k-\eta_k p_j)\phi_i^{\ h}$$
$$-p_t\phi_i^{\ t}(\eta_j\delta_k^{\ h}-\eta_k\delta_j^{\ h}).$$

Transvecting (6.6) with η_h, we find

(6.7) $$\eta_h L_v K_{kji}{}^h=-2\phi_{kj}p_i-\phi_{ki}p_j+\phi_{ji}p_k+p_t\phi_j^{\ t}\gamma_{ki}-p_t\phi_k^{\ t}\gamma_{ji}.$$

Taking the Lie derivative of the both sides of (6.2) we obtain.

$$\eta_t L_v K_{kji}{}^t+(L_v\eta_t)K_{kji}{}^t=(L_v\eta_k)g_{ji}-(L_v\eta_j)g_{ki}$$
$$+\eta_k L_v g_{ji}-\eta_j L_v g_{ki}.$$

Substituting (6.7) into this equation, we obtain

(6.8) $$(L_v\eta_t)K_{kji}{}^t=2\phi_{kj}p_i+\phi_{ki}p_j-\phi_{ji}p_k-p_t\phi_j^{\ t}\gamma_{ki}$$
$$+p_t\phi_k^{\ t}\gamma_{ji}+(L_v\eta_k)g_{ji}+\eta_k L_v g_{ji}$$
$$-(L_v\eta_j)g_{ki}-\eta_j L_v g_{ki}.$$

Transvecting (6.8) with η^k and taking account of (6.2), we obtain

(6.9) $$L_v g_{ji}=p_t\eta^t\phi_{ji}+\eta_j\eta^k L_v g_{ki}.$$

Taking account of the symmetry of (6.9), we obtain

(6.10) $$2p_t\eta^t\phi_{ji}+\eta_j\eta^k L_v g_{ki}-\eta_i\eta^k L_v g_{kj}=0.$$

Transvating (6.10) with η^i, we find

(6.11) $$\eta^k L_v g_{kj}=\tau\eta_j,$$

where we have put

(6.12) $$\tau=\eta^k\eta^i L_v g_{ki}.$$

Substituting (6.11) into (6.9), we see that

(6.13) $$L_v g_{ji}=(p_t\eta^t)\phi_{ji}+\tau\eta_j\eta_i.$$

Taking account of the symmetry of $L_v g_{ji}$, we obtain

(6.14) $$p_t\eta^t=0$$

and from which

(6.15) $$L_v g_{ji}=\tau\eta_j\eta_i$$

by virtue of (6.13).

Operating ∇_k to (6.15), we find

$$\nabla_k(\nabla_j v_i+\nabla_i v_j)=(\nabla_k \tau)\eta_j\eta_i+\tau(\phi_{kj}\eta_i+\phi_{ki}\eta_j). \tag{6.16}$$

Substituting (6.4) into (6.16) and transvecting the result with $\eta^j\eta^i$, we obtain $\nabla_k\tau=0$, that is,

$$\tau=\text{const.} \tag{6.17}$$

On the other hand, substituting (6.14) into the identity;

$$L_v\{{}^{\,h}_{j\ i}\}=\frac{1}{2}g^{ht}(\nabla_j L_v g_{ti}+\nabla_i L_v g_{jt}-\nabla_t L_v g_{ji}),$$

and taking account of (6.17), we obtain

$$L_v\{{}^{\,h}_{j\ i}\}=\tau(\phi_j{}^h\eta_i+\phi_i{}^h\eta_j). \tag{6.18}$$

Comparing (6.4) with (6.18), we obtain

$$p_k\gamma_j{}^h+p_j\gamma_k{}^h-p_t(\phi_k{}^t\phi_j{}^h+\phi_j{}^t\phi_k{}^h)=\tau(\phi_k{}^h\eta_j+\phi_j{}^h\eta_k). \tag{6.19}$$

Transvecting (6.19) with η^j and taking account of (6.14), we easily see that

$$\tau=0$$

and from which

$$p_k\gamma_j{}^h+p_j\gamma_k{}^h-p_t(\phi_k{}^t\phi_j{}^h+\phi_j{}^t\phi_k{}^h)=0.$$

Contracting by $h=j$, we see that

$$p_j=0, \qquad L_v\{{}^{\,h}_{k\ j}\}=0.$$

Thus we have the following

THEOREM 6.1. *In a Sasakian manifold an η-HP vector field with an associated vector other than the zero vector does not exist.*

Sung Kyun Kwan University

REFERENCES

[1] S.-S. Eum, *Cosymplectic manifolds of constant ϕ-holomorphic sectional curvature*, Bull. of Korean Math. Soc., 9 (1972), 1—7.

[2] ________, *On characters of η-related tensors in cosymplectic and Sasakian manifolds* (2), To appear in Jour. of Korean Math. Soc., **21** (1984).

[3] S.-S. Eum and U.K. Kim, *On a decomposition of contravariant C*-analytic vector fields in a compact cosymplectic η-Einstein manifold*, Jour. of Korean Math. Soc., **18**

(1981), 63—71.

[4] S. Sasaki, *Almost contact manifold* I, Lecture note.

[5] K. Yano and H. Hiramatu, *Isometry of Kaehlerian manifolds to complex projective spaces*, Jour. of Math. Soc. of Japan, **33** (1981), 67—78.

[6] K. Yano, *Differential geometry on complex and almost complex spaces*, 1965, Pergamon Press.