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Weak Differentiability and Strict Convexity on Generalized Semi-inner-product Spaces

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1. Introduction

In 1961, G. Lumer ([8]) constructed on a vector space a type of inner product with axiom system more general than that of Hilbert space with the aim of carrying over Hilbert space type arguments to the theory of Banach spaces. There he has considered semi-inner-product space, say, vector space on which, instead of a bilinear form, there is defined a form $[x,y]$ which is linear in one component only, strictly positive and satisfies a Schwarz's inequality.

Later, B. Nath ([9]) has given a straight forward generalization of a semi-inner-product space, called generalized semi-inner-product space, by replaceing a Schwarz's inequality by a Hölder's inequality.

Let X be a complex vector space. A generalized semi-inner-product (in short g.s.i.p.) on X is a complex function $[x,y]$ on $X\times X$ with the following properties:

(1) $[x+y,z]=[x,z]+[y,z]$

(2) $[\lambda x,y]=\lambda[x,y]$

(3) $[x,x]>0$ for $x\neq 0$

(4) $|[x,y]|\leq[x,x]^{\frac{1}{p}}[y,y]^{1-\frac{1}{p}}$, $1<p<\infty$

for all x,y,z in X and for all complex λ.

A vector space with a g.s.i.p. is called a generalized semi-inner-product space (in short g.s.i.p. space). A g.s.i.p. space is a normed vector space with $\|x\|=[x,x]^{\frac{1}{p}}$ ([9]). The topology on a g.s.i.p. space is the one induced by this norm and it will be in this sense that we shall refer to "bounded linear operators". It is also proved in [9] that every normed linear space can be made into a g.s.i.p. space.

A g.s.i.p. space is said to have the homogeneity property when the g.s.i.p. satisfies

(5) $[x, \lambda y]=|\lambda|^{p-2}\bar{\lambda}[x, y]$

for all x, y in X and for all complex λ.

A g.s.i.p. space with the homogeneity property is called a homogeneous g.s.i.p. space.

In fact, every normed linear space can be represented as a homogeneous g.s.i.p. space ([10]).

The principal purpose of this thesis is to study continuity and strict convexity on g.s.i.p. spaces.

We first shall define the convexity, the weak differentiability and the strict convexity on generalized semi-inner-product spaces.

2. Definitions and main results

Definition 2.1. A continuous g.s.i.p. space is a g.s.i.p. space X with the property:

(6) For every x, y in S,

$$[y, x+\lambda y] \to [y, x] \text{ for all real } \lambda \to 0,$$

where $S=\{x \in X : \|x\|=1\}$.

Equivalently, we see that

$$\operatorname{Re}[y, x+\lambda y] \to \operatorname{Re}[y, x]$$

or

$$\operatorname{Im}[y, x+\lambda y] \to \operatorname{Im}[y, x] \text{ as } \lambda \to 0,$$

where Re z and Im z are the real part and the imaginary part of complex z.

Definition 2.2. For x, y in X, x is orthogonal to y if $[y, x]=0$.

A vector x in X is orthogonal to a subspace N if x is orthogonal to all vectors y in N.

For a normed vector space, R. C. James ([6]) studied the orthogonality relation defined by:

$$x \text{ is orthogonal to } y \text{ if } \|x+\lambda y\| \geq \|x\| \text{ for all complex } \lambda.$$

Definition 2.3. ([7], p. 349) A normed vector space is weakly differentiable if for all x, y in S and real λ,

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\|-\|x\|}{\lambda} \text{ exists.}$$

Definition 2.4. A g.s.i.p. space X is strictly convex if whenever $\|x\|+\|y\|=\|x+y\|$ where $x, y \neq 0$, then $y=\lambda x$ for some real $\lambda>0$.

Proposition 2.5. In a continuous g.s.i.p. space with homogeneity, x is orthogonal to y if and only if $\|x+\lambda y\|\geq\|x\|$ for all complex λ.

Proof. If x is orthogonal to y, then

$$\begin{aligned}\|x+\lambda y\|\ \|x\|^{p-1} &\geq |[x+\lambda y, x]| \\ &= |\ \|x\|^p+\lambda[y,x]| \\ &= \|x\|^p.\end{aligned}$$

Conversely, if $\|x+\lambda y\|-\|x\|\geq 0$ for all complex λ, then

$$\mathrm{Re}[x, x+\lambda y]+\mathrm{Re}\lambda[y, x+\lambda y]-|[x, x+\lambda y]|\geq 0,$$

which implies that

$$\mathrm{Re}\ \lambda[y, x+\lambda y]\geq 0$$

for all complex λ.

Therefore for all real λ,

$$\begin{aligned}\mathrm{Re}[y, x+\lambda y] &\geq 0 \text{ for } \lambda\geq 0 \\ &< 0 \text{ for } \lambda< 0.\end{aligned}$$

By the continuity condition, we have for real λ,

$$\mathrm{Re}[y, x+\lambda y]\rightarrow \mathrm{Re}[y, x]$$

Therefore $\mathrm{Re}[y,x]=0$.

For imaginary λ, say $i\lambda_1$ where λ_1 real,

$$\mathrm{Re}\ \lambda[y, x+\lambda y]=\lambda_1\ \mathrm{Re}[iy, x+\lambda_1 iy]\geq 0$$

and again by continuity condition

$$\mathrm{Re}[iy, x]=0, \text{ i.e., } \mathrm{Im}[y, x]=0.$$

Hence we obtain $[y,x]=0$.

Theorem 2.6. A g.s.i.p. space is a continuous g.s.i.p. space if and only if the norm is weakly differentiable.

Proof. Let X be a g.s.i.p. space.

For $x, y\in S$ and real $\lambda>0$,

$$\begin{aligned}\frac{\|x+\lambda y\|-\|x\|}{\lambda} &= \frac{|[x+\lambda y, x]|-\|x\|^p}{\lambda\|x\|^{p-1}} \\ &\geq \frac{\mathrm{Re}[x+\lambda y, x]-\|x\|^p}{\lambda\|x\|^{p-1}} \\ &= \frac{\mathrm{Re}([x, x]+\lambda[y, x])-\|x\|^p}{\lambda\|x\|^{p-1}} \\ &= \frac{\mathrm{Re}[y, x]}{\|x\|^{p-1}}. \qquad \text{(i)}\end{aligned}$$

But also

$$\frac{\|x+\lambda y\|-\|x\|}{\lambda} \leq \frac{\|x+\lambda y\|^p - |[x, x+\lambda y]|}{\lambda \|x+\lambda y\|^{p-1}}$$

(ii) $$= \frac{\mathrm{Re}[y, x+\lambda y]}{\|x+\lambda y\|^{p-1}}$$

Inequality (i) and (ii) show that the continuity property (6) implies that the norm is weakly differentiable.

Conversely refer to [5].

Proposition 2.7. Let X be a g.s.i.p. space. If $\|x\|+\|y\|=\|x+y\|$ on X, then

$$[x, x+y]=\|x\|\ \|x+y\|^{p-1} \text{ and } [y, x+y]=\|y\|\ \|x+y\|^{p-1}.$$

Proof. It is evident for either $x=0$ or $y=0$. We assume that x and y are nonzero elements in X. Then we have

$$[x+y, x+y]=\|x+y\|^p=(\|x\|+\|y\|)(\|x+y\|^{p-1})$$

and so

$$(\|x\|\ \|x+y\|^{p-1}-\mathrm{Re}[x, x+y])+(\|y\|\ \|x+y\|^{p-1}-\mathrm{Re}[y, x+y])=0.$$

Therefore we have

$$[x, x+y]=\|x\|\ \|x+y\|^{p-1}$$

and

$$[y, x+y]=\|y\|\ \|x+y\|^{p-1}.$$

Finally we conclude this paper with useful characterizations of strict convexity on g.s.i.p. space

Theorem 2.8. Let X be a g.s.i.p. space. Then the following statements are equivalent:

(1) X is strictly convex

(2) If $[x, y]=\|x\|\ \|y\|^{p-1}$ where $x, y\neq 0$, then $y=\lambda x$ for some real $\lambda>0$.

(3) If $\|y+z\|=\|y\|$ and $[z, y]=0$, then $z=0$.

Proof. (1)⇒(2).

Let $[x, y]=\|x\|\ \|y\|^{p-1}$, where $x, y\neq 0$. Since

$$\|x\|\ \|y\|^{p-1}+\|y\|^p=[x, y]+[y, y]$$
$$\leq \|x\|\ \|y\|^{p-1}+\|y\|^p,$$

Hence $\|x+y\|=\|x\|+\|y\|$.

So $y=\lambda x$ for some real $\lambda>0$ by (1).

(2)⇒(3).

Let $\|y+z\|=\|y\|$ and $[z, y]=0$. Then

$$[y+z, y]=[y, y]+[z, y]$$
$$=\|y+z\|\ \|y\|^{p-1}.$$

By (2), $y=\lambda(y+z)$ for some real $\lambda>0$. Since $\|y+z\|=\|y\|$, $\lambda=1$.

Therefore $z=0$.

(3)⇒(1).

Let $\|x+y\|=\|x\|+\|y\|$, where $x, y\neq 0$.

By Proposition 2.7,

letting $\lambda=\frac{\|y\|}{\|x\|}$, $z=\lambda x-y$ and $w=x+y$,

then $y=\lambda x$ for some real $\lambda>0$ by (3).

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