HONAM MATHEMATICAL JOURNAL
Volume 6, Number 1, August. 1984

Orthogonal Projections and Unitary Representations of Lie groups

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1. Introduction

This paper is primarily dealt with the study of orthogonality relations for Lie groups based on Haar integral and representation theory for Lie groups, which are considerable parts in Harmonic Analysis. Harmonic Analysis is the study of functions and function spaces defined over a topological group. It has been remarkably developed, since approximately 1970, together with extensive researches on the Haar integral defined over Lie groups ([1], [3], [4]), on the representations of Lie groups ([2], [7], [8], [13]), and on the structure of a Lie group applying Haar integral and representation theory ([11], [12], [14], [16]).

Section 2 presents the general theory of Haar integral which is a radical background for the contents of section 3 and 4. We also derive the Haar measure μ on a Lie group G to get a Hilbert space $L^2(G, \mu)$ that will be used throughout the paper.

In section 3 we develop the elementary representation theory and prove some results on the unitary representations of Lie groups. It is particularly observed that if given a unitary representation $(\pi, (H, \langle\ ,\ \rangle))$ of a Lie group G, we can construct a new unitary representation $(\tilde{\pi}, (\overline{V}, (\ ,\ )))$ by using a positive definite function $\varphi : G \longrightarrow C$. Section 3 also contains the descriptions of Peter-Weyl Theorem and various principal subjects on the representations that will be needed in section 4.

In section 4 we deal with the orthogonal projections of $(\tau, L^2(G))$ to $(\pi_r^* \hat{\otimes} \pi_r, V_r^* \otimes V_r)$ in connection with the Haar integral, and we prove some results concerning the orthogonality relations for compact Lie groups and the characters of finite dimensional irreducible unitary representations of compact Lie groups.

2. Haar Integral

Let M be a connected differentiable (real) manifold with dimension n. For each $m\in M$ we shall denote the tangent space of M at m by M_m. The dual space of M_m will be denoted by M_m^*. We consider the exterior n-bundle $\Lambda_n^*(M)$ of M. The "o-section" of $\Lambda_n^*(M)$ is

$$O=\bigcup_{m\in M}\{0\in\Lambda_n(M_m^*)\}.$$

Since $\Lambda_n(M_m^*)$ is a vector space with dimension 1, each $\Lambda_n(M_m^*)-\{0\}$ has at most two components. Therefore, it follows that $\Lambda_n^*(M)-O$ has at most two components. M is said to be *orientable* if $\Lambda_n^*(M)-O$ has two components. If M is orientable, an *orientation* on M is a choice of one of the two components of $\Lambda_n^*(M)-O$.

A non-connected manifold N is said to be orientable if each component of N is orientable, and an orientation is a choice of orientation of each component.

Proposition 2.1. For a differentiable manifold M of dimension n the following are equivalent.

(i) M is orientable.

(ii) There is a subset $\Phi=\{(V,\psi)\}$ of the atlas of M such that

$$M=\bigcup_{(V,\psi)\in\Phi}V \text{ and } \det\left(\frac{\partial x_i}{\partial y_j}\right)>0 \text{ on } U\cap V,$$

where $(U;\ x_1,\cdots,x_n)$ and $(V;\ y_1,\cdots,y_n)$ are in Φ.

(iii) There is a nowhere vanishing n-form on M.

(See [16] for a proof.)

A Lie group G is a differentiable manifold which is also endowed with a group structure such that the map $G\times G\longrightarrow G$ defined by $(\sigma,\tau)\longmapsto\sigma\tau^{-1}$ is C^∞. In sequel, we shall use G as a Lie group of dimension n without any statements.

The left translation by an element σ of G is the map $L_\sigma\colon G\longrightarrow G$ which is defined by $L_\sigma(\tau)=\sigma\tau$ for each $\tau\in G$. Then L_σ is a diffeomorphism and orientation preserving. Under our situation we have the commutative diagram

$$\begin{array}{ccc}
\Lambda_n^*(G) & \xleftarrow{\ \delta L_\sigma\ } & \Lambda_n^*(G) \\
\downarrow & \copyright & \downarrow \\
G & \xrightarrow{\ L_\sigma\ } & G
\end{array}$$

such that for each $\tau\in G$

$$\delta L_\sigma|_{\sigma\tau}:\ \Lambda_n(G_{\sigma\tau}^*)\longrightarrow\Lambda_n(G_\tau^*).$$

Similarly, the right translation R_σ by an element σ of G can be defined. Of course, R_σ is also a diffeomorphism.

A differential form ω on G is said to be *left invariant* if $\delta L_\sigma\omega=\omega$. Also in general, a nowhere vanishing left invariant form ω is called a *volume form*.

Proposition 2.2. Every Lie group G of dimension n is orientable.

Proof. It is well known that every Lie group G has its Lie algebra $\mathscr{G}$ which is a n-dimensional vector space. Let $E_l{}^1{}_{inv}(G)$ be the set of all left invariant 1-forms of G. Then we see that $\mathscr{G}^*\cong E_l{}^1{}_{inv}(G)$ ([16]). Acoordingly, there is a basis $\{\omega_1,\cdots,\omega_n\}$ of $E_l{}^1{}_{inv}(G)$. In this case, $\omega_1\wedge\omega_2\wedge\cdots\wedge\omega_n$ is a nowhere vanishing n-form of G. Thus by Proposition 2.1, G is orientable. ■

Let G be a n-dimensional Lie group. Then G is orientable as above, and $\Lambda_n(G_\sigma{}^*)(\sigma\in G)$ is an 1-dimensional vector space. Choose a non-zero left invariant n-form ω consistent with the fixed orientation on G. We define, with respect to ω, the *integral* of compactly supported continuous function f from G into C by setting

$$\int_G f=\int_G f\omega, \tag{2.1}$$

that is, for each point $x\in G$ and its local coordinate system $(U;\ x_1,\ \cdots,\ x_n)$

$$\begin{aligned}\int_U f&=\int_U f(x)\,dx_1\wedge\cdots\wedge dx_n\\ &=\int_{a_n}^{b_n}\cdots\cdots\int_{a_1}^{b_1}f(x)\,dx_1\cdots\cdots dx_n,\end{aligned}$$

where $U=\{x=(x_1,\ \cdots,\ x_n)\,|\,a_i<x_i<b_i,\ i=1,\ \cdots,\ n\}$. This integral depends on the choice of the non-zero left invariant n-form ω consistent with the orientation on G.

Since such forms are uniquely determined up to a positive constant multiple, so is the integral. If G is a compact Lie group of dimension n, then we can and always will fix the choice of ω by requiring the normalization

$$\int_G\omega=1. \tag{2.2}$$

Since a volume form ω is left invariant (i. e. $^\forall\sigma\in G,\ \delta L_\sigma(\omega)=\omega$), in the integral (2.1) we have

$$\int_G\delta L_\sigma(f\omega)=\int_G(f\circ L_\sigma)\delta L_\sigma(\omega)=\int_G f\circ L_\sigma\omega=\int_G f\circ L_\sigma.$$

That is, for each $\sigma\in G$,

$$\int_G f = \int_G f \circ L_\sigma. \tag{2.3}$$

In view of property (2.3) the integral (2.1) is called a *left invariant integral.*

Proposition 2.3. In the integral (2.1) the following holds.

(i) For each $\sigma \in G$ and a compactly supported continuous function f from G into $\boldsymbol{C}$

$$\int_G f\omega = \int_G (f \circ R_\sigma)\, \lambda(\sigma)\, \omega,$$

where $\lambda: G \longrightarrow \boldsymbol{R}^+$ ($\boldsymbol{R}^+ = \{r \in \boldsymbol{R} \mid r > 0\}$ is a multiplicative Lie group) is a Lie group homomorphism.

(ii) Let $\eta: G \longrightarrow G$ be defined by $\sigma \longmapsto \sigma^{-1}$. Then, with the situation (i), we have

$$\int_G \delta\eta(f\omega) = \int_G f \circ \eta \frac{\omega}{\lambda(\sigma)}.$$

Proof. (i) We first note that, for each $\sigma, \tau \in G$, $L_\sigma \circ R_\tau = R_\tau \circ L_\sigma$.
Then we have

$$\delta(R_\tau \circ L_\sigma)\omega = \delta L_\sigma \circ \delta R_\tau \omega, \text{ and}$$
$$\delta(R_\tau \circ L_\sigma)\omega = \delta(L_\sigma \circ R_\tau)\omega = \delta R_\tau \circ \delta L_\sigma \omega = \delta R_\tau \omega.$$

That is, $\delta R_\tau \omega$ is left invariant, and hence there exists a non-zero real number $\bar{\lambda}(\tau)$ such that

$$\delta R_\tau \omega = \bar{\lambda}(\tau)\omega.$$

Let $(U;\ x_1, \cdots, x_n)$ and $(V;\ y_1, \cdots, y_n)$ be two local coordinate systems of $\tau \in G$ such that

$$\omega = dx_1 \wedge \cdots\cdots \wedge dx_n, \qquad \delta R_\tau \omega = dy_1 \wedge \cdots\cdots \wedge dy_n.$$

Then we have

$$dy_1 \wedge \cdots\cdots \wedge dy_n = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots\cdots & \frac{\partial y_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \cdots\cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} dx_1 \wedge \cdots\cdots \wedge dx_n,$$

and thus

$$\bar{\lambda}(\tau)=\begin{vmatrix} \frac{\partial y_1}{\partial x_1} \cdots\cdots \frac{\partial y_1}{\partial x_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial y_n}{\partial x_1} \cdots\cdots \frac{\partial y_n}{\partial x_n} \end{vmatrix} .$$

Therefore $\bar{\lambda}$ is a C^∞−function (Note that if $\delta R_\tau \omega$ and ω have the same orientation $\bar{\lambda}(\tau)>0$, otherwise $\bar{\lambda}(\tau)<0$). We put

$$\lambda(\tau)=|\bar{\lambda}(\tau)|,$$

then λ is also a C^∞-function. Since $R_{\sigma\tau}=R_\tau \circ R_\sigma$ for each $\sigma,\tau \in G$ we have

$$\begin{aligned}\bar{\lambda}(\sigma\tau)\omega &= \delta R_{\sigma\tau}\omega=\delta(R_\tau \circ R_\sigma)\omega \\ &= \delta R_\sigma \circ \delta R_\tau \omega = \delta R_\sigma \bar{\lambda}(\tau)\omega \\ &= \bar{\lambda}(\sigma)\bar{\lambda}(\tau)\omega.\end{aligned}$$

This means that

$$\lambda(\sigma\tau)=\lambda(\sigma)\lambda(\tau).$$

Therefore $\lambda: G \longrightarrow \boldsymbol{R}^+$ is a Lie group homomorphism.
Now for a diffeomorphism

$$\gamma: G \longrightarrow G$$

we have

$$\int_G \omega = \pm \int_G \delta\gamma(\omega)$$

with "+" if and only if γ is orientation−preserving.
Therefore, in our context

$$\begin{aligned}\int_G f\omega &= \pm \int_G \delta R_\sigma(f\omega) \\ &= \pm \int_G (f\circ R_\sigma)\bar{\lambda}(\sigma)\omega \\ &= \int_G (f\circ R_\sigma)(\lambda(\sigma))\omega.\end{aligned}$$

(ii) We note that $L_\sigma \circ \eta = \eta \circ R_{\sigma^{-1}}$ $(\sigma \in G)$ implies that

$$\delta(L_\sigma \circ \eta)\omega = \delta(\eta \circ R_{\sigma^{-1}})\omega.$$

And thus we have

$$\delta\eta\circ\delta L_{\sigma}\omega=\delta\eta\omega=\delta R_{\sigma^{-1}}\circ\ \delta\eta\omega,$$

that is, $\delta\eta\omega$ is a right invariant n-form. Since there is a relation $\omega^R=d\omega^L$ between a left invariant n-form ω^L and a right invariant n-form ω^R ([15]), where d is a nowhere vanishing C^∞ real function on G, we have

$$\delta\eta\omega=d\omega.$$

Since for each $\sigma\in G$

$$\begin{aligned}\delta R_{\sigma}(\delta\eta\omega)&=\delta\eta\omega\\&=\delta R_{\sigma}(d(x\sigma)\omega)\\&=\bar{\lambda}(\sigma)\ d(x\sigma)\omega\quad({}^{\forall}x\in G),\end{aligned}$$

we have

$$\bar{\lambda}(\sigma)=d(x)(d(x\sigma))^{-1}\text{ for all }x\in G.$$

Noting that $d(e)=1$, we see that

$$d=\frac{1}{\bar{\lambda}(\sigma)}.$$

Moreover, since η is a diffeomorphism,

$$\begin{aligned}\int_G f\omega&=\pm\int_G\delta\eta(f\omega)\\&=\pm\int_G(f\circ\eta)\frac{1}{\bar{\lambda}(\sigma)}\omega=\int_G(f\circ\eta)\frac{\omega}{\lambda(\sigma)}\end{aligned}\quad\blacksquare$$

The function λ: $G\longrightarrow R^+$ in the proof above is called the *modular function* of G.

Proposition 2.4. For a compact Lie group G the integral (2.1) is left and right invariant.

Proof. It has been shown that the integral (2.1) is left invariant.
By (2.2) above, for the function $f: G\longrightarrow C$ such that $f\equiv 1$ we have

$$1=\int_G f\omega=\int_G(f\circ R_{\sigma})\ \lambda(\sigma)\,\omega=\lambda(\sigma)\int_G f\omega$$

since $f\circ R_{\sigma}=f\equiv 1$. Therefore $\lambda(\sigma)=1$. As in the proof of Proposition 2.3, since

$$\int_G f\omega = \int_G (f\circ R_\sigma)(\lambda(\sigma))\ \omega$$
$$= \int_G f\circ R_\sigma \omega = \int_G f\circ R_\sigma,$$

the integral (2.1) is right invariant. ■

A Lie group G for which $\lambda\equiv 1$ is called *unimodular*.
Thus, every compact Lie group with finite dimension is unimodular.

For a Lie group G we put

$$C_0(G)=\{f : G\longrightarrow C\,|\,f \text{ is continuous and the support of } f \text{ is compact}\}.$$

We note that for $f\in C_0(G)$ such that $f\not\equiv 0$ and $f\geq 0$

$$\int_G f\omega > 0 \tag{2.4}$$

since there is a positive coordinate patch for G on which f is positive.
Therefore we obtain ([6] or [15])

$$\int_G f\omega = 0 \text{ if and only if } f\equiv 0. \tag{2.5}$$

Furthermore, we also get ([10])

$$\left|\int_G f\omega\right| \leq \int_G |f|\omega. \tag{2.6}$$

Definition 2.5. A measure μ on a locally compact Hausdorff space X is a linear function $\mu : C_0(X)\longrightarrow C$ such that if for each compact subset $K\subset X$ and for each $f\in C_0(X)$ with Supp$f\subset K$ there exists a constant N_K such that

$$|\mu(f)|\leq N_K\|f\|_\infty,$$

where $\|f\|_\infty = \sup_{x\in X}|f(x)|$.
A measure μ is said to be *positive* if $f\geq 0$ then $\mu(f)\geq 0$.
A positive measure μ is *normal* if for each $f\in C_0(X)$ with $f\geq 0$ and $f\not\equiv 0$, $\mu(f)>0$.

For a Lie group G we have already defined the left invariant integral $\int_G f\omega$ for $f\in C_0(G)$. We define

$$\mu : C_0(G)\longrightarrow C \quad \text{by} \quad \mu(f)=\int_G f\omega.$$

Then it is easy to prove that μ is a positive and normal measure on G (by (2.4) and (2.6)).

This measure μ is called *Haar measure* and the integral (2.1) is called *Haar integral.*
For $f_1, f_2 \in C_0(G)$ we define

$$\langle f_1, f_2 \rangle = \mu(f_1 \bar{f_2}) = \int_G f_1 \bar{f_2}\, \omega.$$

Then it is clear that $(C_0(G), \langle , \rangle)$ is a pre-Hilbert space.
Let $L^2(G, \mu)$ be the completion of $(C_0(G), \langle , \rangle)$. That is, $L^2(G, \mu)$ is a Hilbert space and $\overline{(C_0(G), \langle , \rangle)} = L^2(G, \mu)$.

Definition 2.6. A Hilbert space is said to be separable if it contains a countable dense subset.

Proposition 2.7. For each compact Lie group G, $L^2(G, \mu)$ is separable, where μ is Haar measure on G.

Proof. Since G is compact there is a set $\{(U_1, \varphi_1), \cdots\cdots, (U_n, \varphi_n)\}$ of local coordinate systems such that $G = \bigcup_{i=1}^{n} U_i$. Since G is a Lie group there exists a partition of unity $\{\psi_j \mid j=1, \cdots, n\}$ such that

$$\mathrm{Supp}\psi_j \subset U_j,\ \psi_j \geq 0 \text{ and } \sum_{j=1}^{n} \psi_j(x) = 1 \quad (x \in G).$$

Suppose that $f \in C_0(G)$ is such that $\langle f, \psi_j \rangle = 0$ for all $j=1, 2, \cdots, n$. Then $\langle f, \psi_j \rangle = 0$ $\rightleftarrows \langle Ref, \psi_j \rangle = 0 = \langle Im\ f, \psi_j \rangle$, and therefore we may suppose that f is real. Now assume that $f(x_0) \neq 0$ for some $x_0 \in G$. Then there is a connected open neighborhood U_0 of x_0 such that $f(x) \neq 0$ for all $x \in U_0$.
Therefore $f|U_0 > 0$ or $f|U_0 < 0$. By possibly substituting $-f$ for f we assume that $f|U_0 > 0$. Let ψ_j be so that $\psi_j(x_0) > 0$ and $\mathrm{Supp}\psi_j \subset U_0$. Then $\psi_j f \in C_0(G)$ and by (2.4) and (2.5)

$$\mu(\psi_j f) = \int_G \psi_j f \omega > 0.$$

This is a contradiction to our hypothesis. Therefore $f \equiv 0$.
We put

$$\psi = \psi_1 + \cdots\cdots + \psi_n.$$

and

$$\varphi = \frac{\psi}{\mu(\psi^2)}\ .$$

Then $\{\varphi\}$ is an orthonormal basis of $L^2(G, \mu)$. Since there exists an orthonormal basis in $L^2(G, \mu)$ if and only if $L^2(G, \mu)$ is separable ([15]), our proof is complete. ■

3. Unitary Representations

Let $(H, \langle\ ,\ \rangle)$ be a complex Hilbert space. A *representation* of a Lie group G on H is a homomorphism

$$\pi : G \longrightarrow \mathrm{Aut}(H)$$

such that the map

$$\begin{array}{ccc} G\times H & \longrightarrow & H \\ \Cup & & \Cup \\ (g, v) & \longmapsto & \pi(g)v \end{array}$$

is continuous, where $\mathrm{Aut}(H)$ is the group consisting of all automorphisms of H. In this case, (π, H) denotes the representation of G on H.

(π, H) is said to be unitary if for each $g\in G$, $\pi(g)$ is a unitary operator. Since, if $\pi(g)$ is unitary then for $v, w\in H$

$$\langle v, w\rangle = \langle \pi(g)v, \ \pi(g)w\rangle,$$

we have that

$$(\pi, H) \text{ is unitary } \rightleftarrows {}^{\forall}g\in G,\ \pi(g)^*\pi(g)=\pi(g)\pi(g)^*=I,$$

where $\pi(g)^*$ is the adjoint operator of $\pi(g)$ and I is the identity map of H. We note that $\pi(g)^* = {}^t\overline{\pi(g)}$ if H is a finite dimensional complex vector space, where $\overline{\pi(g)}$ is the conjugate of $\pi(g)$ and ${}^t\pi(g)$ is the transpose of $\pi(g)$.

Let (π, H) be a representation of G on H. A subspace W of H is said to be *invariant* if for each $g\in G$, $\pi(g)(W)\subset W$. (π, H) is said to be *irreducible* if the only closed invariant subspaces of H are H and $\{0\}$.

For two representations (π, H) and (ρ, V) of G we put $\mathrm{Hom}_G(H, V) = \{A\colon H \longrightarrow V \mid A$ is linear, continuous, and $A\pi(x)=\rho(x)A$ for all $x\in G\}$.
If there is an isomorphism in $\mathrm{Hom}_G(H, V)$ then (π, H) and (ρ, V) are said to be *equivalent*.

Definition 3.1. Let $\{(\pi_i, H_i)\}$ be a countable collection of representations of G. A representation (π, H) of G is said to be a *direct sum* of the (π_i, H_i) if for each i there is an injective element A_i in $\mathrm{Hom}_G(H_i, H)$ such that

(i) the sum $V=\sum_i A_i(H_i)$ is direct

(ii) V is a dense subspace of H.

In this case we write $H=\sum H_i$ as a direct sum.

Definition 3.2. For two unitary representations (π, H) and (ρ, V) of G if there is an isomorphism in $\mathrm{Hom}_G(H, V)$ then they are said to be *unitarily equivalent*. (π, H) is a *unitary direct sum* of (π_i, H_i) if

(i) $H=\sum H_i$

(ii) for each i (π_i, H_i) is a unitary representation of G

(iii) when we identify H_i with its image in H, H_i are mutually orthogonal.

Let G be a Lie group. A continuous map $\varphi: G \longrightarrow C$ is *positive definite* if for any subsets $\{c_1, \cdots, c_n\} \subset C$ and $\{g_1, \cdots, g_n\} \subset G$

$$\sum_{i,j=1}^{n} c_i \bar{c}_j \varphi(g_j^{-1} g_i) \geq 0$$

Theorem 3.3. Let $(\pi, (H, \langle \ , \ \rangle))$ be a unitary representation of G.

(i) Define $\varphi: G \longrightarrow C$ by $\varphi(g)=\langle \pi(g)v, v\rangle$ for $g \in G$ and a fixed element v of H. Then φ is positive definite.

(ii) For a positive definite function $\varphi: G \longrightarrow C$, let V be the vector space which is linearly generated by the set $\{f: G \longrightarrow C \mid f$ is continuous and such that for all $g \in G, f(g) =\varphi(gx)$ with a fixed element x in $G\}$.

(a) For $f, h \in V$ with

$$f(g)=\sum_{i=1}^{n} c_i \varphi(gx_i), \qquad h(g)=\sum_{j=1}^{m} d_j \varphi(gy_j)$$

define

$$(f, h)=\sum_{\substack{i=1,\cdots,n \\ j=1,\cdots,m}} c_i \bar{d}_j \varphi(y_j^{-1} x_i).$$

If φ satisfies that for any subsets $\{c_1, \cdots, c_n\} \subset C$ and $\{g_1, \cdots, g_n\} \subset G$

$$\sum_{i,j=1}^{n} c_i \bar{c}_j \varphi(g_j^{-1} g_i)=0 \Longrightarrow \sum_{i,j=1}^{n} c_i \bar{c}_j \varphi(g_j^{-1}) \varphi(g_i)=0,$$

then $(\ ,\ )$ is an inner product of V.

(b) Let $(\bar{V}, (\ ,\ ))$ be the completion of $(V, (\ ,\ ))$.

If we define $\bar{\pi}(g) f(x)=f(xg)$ for $x, g \in G$ and $f \in \bar{V}$, then $(\bar{\pi}, \bar{V})$ is a unitary representation of G.

(c) $(f-f_1, h)=0$ for all $h\in\bar{V}$ if and only if $f=f_1$, where f and f_1 are in $\bar{V}$.

Proof. (i) For $\{c_1, \cdots, c_n\}\subset C$ and $\{g_1, \cdots, g_n\}\subset G$,

$$\begin{aligned}\sum_{i,j=1}^{n} c_i \bar{c}_j\ \varphi(g_j^{-1} g_i) &= \sum_{i,j=1}^{n} c_i\ \bar{c}_j\langle\pi(g_j^{-1} g_i)v,\ v\rangle\\ &= \sum_{i,j=1}^{n} c_i \bar{c}_j\langle\pi(g_j)^{-1}\pi(g_i)\,v,\ v\rangle\\ &= \sum_{i,j=1}^{n} c_i \bar{c}_j\langle\pi(g_i)\,v,\ \pi(g_j)v\rangle \qquad (\pi(g_j)^{-1}=\pi(g_j)^*)\\ &= \sum_{i,j=1}^{n} \langle c_i\pi(g_i)\,v,\ c_j\pi(g_j)v\rangle\\ &= \langle c_1\pi(g_1)v+\cdots+c_n\pi(g_n)v,\ c_1\pi(g_1)v+\cdots+c_n\pi(g_n)v\rangle\\ &\geq 0.\end{aligned}$$

Thus φ is positive definite.

(a) of (ii). Suppose that we have three elements f_1, f_2 and h in V such that

$$\left.\begin{aligned} &f_1(g)=\sum_{i=1}^{m} c_i\varphi(gx_i), \qquad f_2(g)=\sum_{j=m+1}^{m+n} c_j\ \varphi(gx_j)\\ &h(g)=\sum_{k=1}^{l} d_k\varphi(gy_k)\end{aligned}\right\} \tag{3.1}$$

Since

$$f_1(g)+f_2(g)=\sum_{i=1}^{m+n} c_i\varphi(gx_i),$$

we have the following:

$$\begin{aligned}(f_1+f_2,\ h) &= \sum_{\substack{i=1,\cdots,m+n\\ k=1,\dots,l}} c_i\bar{d}_k\varphi(y_k^{-1}x_i)\\ &= \sum_{\substack{i=1,\cdots,m\\ k=1,\cdots,l}} c_i\bar{d}_k\varphi(y_k^{-1}x_i)+\sum_{\substack{i=m+1,\cdots,m+n\\ k=1,\cdots,l}} c_i\bar{d}_k\varphi(y_k^{-1}x_i)\\ &=(f_1,\ h)+(f_2,\ h).\end{aligned}$$

Similarly, we can prove that $(f,\ h_1+h_2)=(f,\ h_1)+(f,\ h_2)$.

Next, since φ is positive definite, for all pair $\{\{c_1, \cdots, c_n\}\subset C,\ \{g_1, \cdots\ g_n\}\subset G\}$ we have

$$\begin{aligned}0\leq\sum_{i,j=1}^{n} c_j\bar{c}_i\ \varphi(g_i^{-1}g_j) &= \sum_{i,j=1}^{n} c_i\bar{c}_j\varphi(g_j^{-1}g_i)\\ &= \sum_{i,j=1}^{n}\overline{c_i\bar{c}_j\ \varphi(g_j^{-1}g_i)}\end{aligned}$$

$$=\sum_{i,j=1}^{n} c_j \bar{c}_i \overline{\varphi(g_j^{-1} g_i)}$$

and thus

$$\overline{\varphi(g_j^{-1} g_i)}=\varphi(g_i^{-1} g_j).$$

Hence, it follows that $\overline{\varphi(g)}=\varphi(g^{-1})$

Using notations in (3.1)

$$\overline{(h, f_1)}=\sum_{\substack{i=1,\cdots,m\\k=1,\cdots,l}} \overline{d_k \bar{c}_i \varphi(x_i^{-1} y_k)}$$
$$=\sum_{\substack{i=1,\cdots,m\\k=1,\cdots,l}} c_i \bar{d}_k \varphi(y_k^{-1} x_i)$$
$$=(f_1, h).$$

That is, we have that $(f_1, h)=\overline{(h, f_1)}$

Finally, we have to prove that for $f\in V$

$$(f, f)\geq 0, \text{ and } (f, f)=0 \rightleftarrows f=0$$

For $f(g)=\sum_{i=1}^{n} c_i \varphi(gx_i)$,

$$(f, f)=\sum_{i,j=1}^{n} c_i \bar{c}_j \varphi(x_j^{-1} x_i)\geq 0.$$

Assume that $f=0$. Then

$$(f, f)=\sum_{i,j=1}^{n} c_i \bar{c}_j \varphi(x_j^{-1} x_i)$$
$$=\sum_{j=1}^{n} \bar{c}_j(c_1 \varphi(x_j^{-1} x_1)+\cdots+c_n \varphi(x_j^{-1} x_n))$$
$$=\sum_{j=1}^{n} \bar{c}_j f(x_j^{-1})$$
$$=0.$$

Conversely, we assume that $(f, f)=0$. Recall that

$$f(g)=\sum_{i=1}^{n} c_i \varphi(gx_i) \text{ and } \overline{f(g)}=\sum_{i=1}^{n} \bar{c}_i \varphi(x_i^{-1} g^{-1})$$

for all $g\in G$. Hence we have

$$f(g)\cdot\overline{f(g)}=\sum_{i,j=1}^{n} c_i \bar{c}_j \varphi(X_j^{-1}) \varphi(X_i),$$

where $X_i = gx_i$. Since $X_j^{-1} X_i = x_j^{-1} x_i$,

$$(f, f) = \sum_{i,j=1}^{n} c_i \bar{c}_j \varphi(x_j^{-1} x_i) = \sum_{i,j=1}^{n} c_i \bar{c}_j \varphi(X_j^{-1} X_i) = 0$$

Hence by our assumption we have

$$f(g) \cdot \overline{f(g)} = 0,$$

and thus $f(g) = 0$ for all $g \in G$.

(b) of (ii). Since for $x, g \in G$ and $f \in \bar{V}$, $\bar{\pi}(g) f(x) = f(xg)$, we have

$$\bar{\pi}(g_1)(\bar{\pi}(g_2) f)(x) = \bar{\pi}(g_2) f(xg_1) = f(xg_1 g_2) = \bar{\pi}(g_1 g_2) f(x)$$

for $g_1, g_2 \in G$, and thus

$$\bar{\pi} : G \longrightarrow \mathrm{Aut}(\bar{V})$$

is a group homomorphism.

For $f, h \in \bar{V}$ and $g \in G$ we want to prove that

$$(f, h) = (\bar{\pi}(g) f, \bar{\pi}(g) h).$$

Assume that

$$f(g_1) = \sum_{i=1}^{m} c_i \varphi(g_1 x_i), \qquad h(g_1) = \sum_{j=1}^{n} d_j \varphi(g_1 y_j).$$

Then

$$(\bar{\pi}(g) f)(g_1) = \sum_{i=1}^{m} c_i \varphi(g_1 g x_i), \quad (\bar{\pi}(g) h)(g_1) = \sum_{j=1}^{n} d_j \varphi(g_1 g y_j)$$

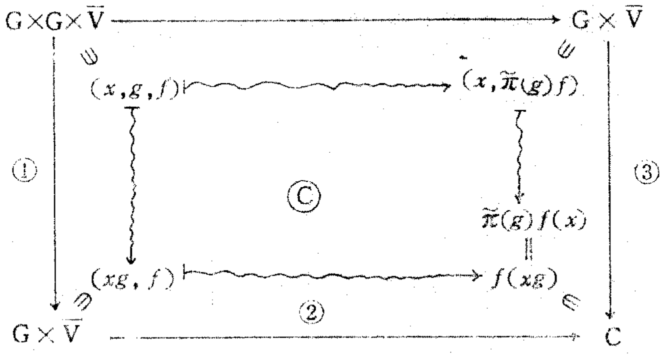
Hence we have the following:

$$\begin{aligned}(f, h) &= \sum_{\substack{i=1,\cdots,m \\ j=1,\cdots,n}} c_i \bar{d}_j \varphi(y_j^{-1} x_i) \\ &= \sum_{\substack{i=1,\cdots,m \\ j=1,\cdots,n}} c_i \bar{d}_j \varphi(y_j^{-1} g^{-1} g x_i) \\ &= (\bar{\pi}(g) f, \bar{\pi}(g) h)\end{aligned}$$

Finally we have to prove that the map

$$G \times \bar{V} \longrightarrow \bar{V} \qquad ((g, f) \longmapsto \bar{\pi}(g) f)$$

is continuous. In the commutative diagram



the maps ①, ② and ③ are continuous. Therefore, the map $(g, f) \longmapsto \tilde{\pi}(g)f$ is continuous.

(c) of (ii). If we use the notations in (3.1) we get

$$(f_1-f_2,\ h)=\sum_{\substack{i=1,\cdots,m\\ j=1,\cdots,l}} c_i \bar{d}_j \varphi(y_j^{-1}x_i)-\sum_{\substack{i=m+1,\cdots,m+n\\ j=1,\cdots,l}} c_i \bar{d}_j \varphi(y_j^{-1}x_i)$$

$$=\sum_{j=1}^{l}\bar{d}_j\{c_1\varphi(y_j^{-1}x_1)+\cdots\cdots+c_m\varphi(y_j^{-1}x_m)\}$$

$$-\sum_{j=1}^{l}\bar{d}_j\{c_{m+1}\varphi(y_j^{-1}x_{m+1})+\cdots\cdots+c_{m+n}\varphi(y_j^{-1}x_{m+n})\}$$

$$=\sum_{j=1}^{l}\bar{d}_j f_1(y_j^{-1})-\sum_{j=1}^{l}\bar{d}_j f_2(y_j^{-1})$$

$$=\sum_{j=1}^{l}\bar{d}_j\{f_1(y_j^{-1})-f_2(y_j^{-1})\}$$

Since $y_j (j=1, \cdots, l)$, $d_j (j=1, \cdots, l)$ and l are arbitrary,

$$(f_1-f_2,\ h)=0 \text{ for all } h\in\bar{V} \text{ if and only if } f_1=f_2. \quad ■$$

A representation (π, H) of a Lie group G is called a *finite dimensional representation* if $\dim_C H<\infty$.

Lemma 3.4. Let (π, H) be a finite dimensional irreducible representation of a Lie group G. Then $\mathrm{Hom}_G(H, H)=CI$.

Proof. Since $\dim_C H<\infty$ and C is an algebraically closed field, each $A\in\mathrm{Hom}_G(H,H)$ has an eigenvector $v(\neq 0)\in H$ with eigenvalue λ. Put

$$H_\lambda=\{w\in H \mid Aw=\lambda w\}.$$

Since for each $g \in G$, $\pi(g)$ satisfies $\pi(g)A = A\,\pi(g)$, we have

$$\pi(g)Aw = \lambda(\pi(g)w) = A(\pi(g)w)$$

for each $w \in H_\lambda$, and hence $\pi(g)w \in H_\lambda$. It follows that $\pi(G)H_\lambda \subset H_\lambda$. Thus by our irreducibility we have

$$H_\lambda = H \text{ and } A = \lambda I.$$

Therefore $\mathrm{Hom}_G(H, H) \subset CI$, and hence $\mathrm{Hom}_G(H, H) = CI$. ■

Let (π, H) be a finite dimensional representation of a Lie group G, and let H^* be the dual space of H.
For an orthonormal basis $\{v_1, \cdots, v_n\}$ $(\dim_C H = n)$ of H, let $\{v_1^*, \cdots, v_n^*\}$ be the dual basis of $\{v_1, \cdots, v_n\}$. Then by setting

$$\langle v_i^*, v_j^* \rangle = \delta_{ij}$$

H^* becomes a Hilbert space. We define for each $g \in G$, $v \in H$

$$(\pi^*(g)v_i^*)(v) = v_i^*(\pi(g)^{-1}v)(i = 1, \cdots, n). \qquad (3.2)$$

Then(π^*, H^*) is also a finite dimensional representation of G.
It is easy to show that (π, H) is unitary and irreducible if and only if (π^*, H^*) is unitary and irreducible.

Definition 3.5. Let (π_1, H_1) and (π_2, H_2) be finite dimensional representation of G. We give

$$H = H_1 \otimes H_2 \ (\otimes = \otimes_C)$$

the Hilbert space structure that makes the basis $\{v_i \otimes w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ orthonormal if $\{v_1, \cdots, v_m\}$ and $\{w_1, \cdots, w_n\}$ are othornomal basis of H_1 and H_2, respectively.
We define

$$\pi : G \longrightarrow \mathrm{Aut}(H_1 \otimes H_2) \text{ by } \pi(g) = \pi_1(g) \otimes \pi_2(g) \text{ for each } g \in G,$$

then (π, H) is a finite dimensional representation of G which is called the *tensor product representation* of H_1 and H_2. Here we note that

$$\pi(g)(v \otimes w) = \pi_1(g)v \otimes \pi_2(g)w \quad (g \in G, \ v \in H_1, \ w \in H_2).$$

Also we define

$$\begin{array}{cccc} \pi_1 \hat{\otimes} \pi_2 : & G \times G & \longrightarrow & \mathrm{Aut}(H_1 \otimes H_2) \\ & \Downarrow & & \Downarrow \\ \text{by} & (g, h) & \longmapsto & \pi_1(g) \otimes \pi_2(h), \end{array}$$

where $(\pi_1(g)\otimes\pi_2(h))(v\otimes w)=\pi_1(g)v\otimes\pi_2(h)w$ for $v\in H_1$, $w\in H_2$.

Then $(\pi_1\hat{\otimes}\pi_2, H_1\otimes H_2)$ is a finite dimensional representation of $G\times G$ which is called the *exterior tensor product representation* of H_1 and H_2.

Lemma 3.6. Let $(\pi, (H, \langle\ ,\ \rangle))$ be a unitary, irreducible, and finite dimensional representation of a Lie group G. Then any invariant Hermitian form on H is a real multiple of $\langle\ ,\ \rangle$.

Proof. If h is an invariant Hermitian form on H then for $v, w\in H$, $h(v,w)=h(\pi(g)v, \pi(g)w)$ $(g\in G)$. There exists a Hermitian matrix A such that $h(v, w)=\langle Av, w\rangle$, and we note that

$$\begin{cases} \langle \pi(g)v, \pi(g)w\rangle = \langle \pi(g)Av, \pi(g)w\rangle \\ \langle \pi(g)v, \pi(g)w\rangle = \langle A\pi(g)v, \pi(g)w\rangle \end{cases}$$

implies that $\pi(g)A=A\pi(g)$, and hence $A\in\mathrm{Hom}_G(H,H)$. Therefore, by our hypothesis and Lemma 3.4, $A=cI$, where $c\in C$. Since $A=A^*={}^t\bar{A}$, c must be a real number. Therefore we have

$$h(v, w)=c\langle v, w\rangle. \quad \blacksquare$$

In sequel, we shall deal with compact Lie groups and finite dimensional representations if there is no any statements. Accordingly, by a representation (π, H) of Lie group G we mean a finite dimensional representation of a compact Lie group without any statements.

As in §2, for a Lie group G, $L^2(G, \mu)$ is a Hilbert space with Haar measure (for simplicity we shall put $L^2(G, \mu)=L^2(G)$).
For $f\in C(G)$ (since G is compact $C_0(G)=C(G)=\{f\colon G\longrightarrow C \mid f \text{ is continuous}\}$) and $g\in G$, we put

$$(\pi(g)f)(x)=f(g^{-1}x) \quad (x\in G) \tag{3.3}$$

Then $(\pi, L^2(G))$ becomes a unitary representation of G (note that $(\pi, L^2(G))$ is not finite dimensional) ([15], [11], [1]). Furthermore, if we define

$$(\tau(g, h)f)(x)=f(g^{-1}xh) \quad (x\in G) \tag{3.4}$$

then $(\tau, L^2(G))$ is a unitary representation of $G\times G$ ([15]).
For a compact Lie group G, let $\tilde{G}$ be the set of all equivalence classes of irreducible, finite dimensional, and unitary representations of G. That is, for each $\gamma\in\tilde{G}$ let (π_γ, V_γ) be a fixed representative. In this case

$$A_\gamma : V_\gamma^*\otimes V_\gamma \longrightarrow C(G)$$

is defined by

$$A_\gamma(\lambda\otimes v)(g)=\lambda(\pi_\gamma(g)v) \tag{3.5}$$

for $\lambda\in V_\gamma^*$, $v\in V_\gamma$ and $g\in G$. Then

$$A_\gamma\in \mathrm{Hom}_{G\times G}((\pi_\gamma^*\hat{\otimes}\pi_\gamma, V_\gamma^*\otimes V_\gamma), (\tau, L^2(G)))$$

(see (3.4) and Definition 3.5).

We need the follwoing theorem in section 4

Theorem 3.7. (Peter-Weyl) Under the above notations, $(\tau, L^2(G))=\sum V_\gamma^*\otimes V_\gamma$ which is a unitary direct sum over $\tilde{G}$ of representations of $G\times G$. In particular, $\tilde{G}$ is countable. (For proofs see [15] or [6]).

We must note in the Peter-Weyl Theorem that for each $\gamma\in G$ the Hilbert space structure of $V_\gamma^*\otimes V_\gamma$ is defined by

$$\langle v^*\otimes v, w^*\otimes w\rangle=\langle v, w\rangle\langle v^*, w^*\rangle \tag{3.6}$$

for v, $w\in V_\gamma$ and v^*, $w^*\in V_\gamma^*$. Of course, the Hilbert space structure of V_γ^* is induced from the structure of V_γ (see (3.2) above).

4. Orthogonal Projections

Let V_1 and V_2 be finite dimensional (complex) vector space. We put

$$L(V_1, V_2)=\{f: V_1\longrightarrow V_2 \mid f \text{ is a linear transformation}\}.$$

We first prove the following.

Proposition 4.1. For finite dimensional vector spaces V_1 and V_2 we have the isomorphism (as vector spaces)

$$\Psi : V_1^*\otimes V_2\cong L(V_1, V_2),$$

where $\otimes = \otimes_C$.

Proof. We consider the map

$$\Psi : V_1^* \otimes V_2 \longrightarrow L(V_1, V_2)$$

defined by

$$\Psi(\lambda \otimes w)(v) = \lambda(v)w$$

for $\lambda \in V_1^*$, $v \in V_1$ and $w \in V_2$. It is clear that Ψ is a bilinear map. Let us define the inverse Φ of Ψ as follows.

We put

$\{v_1, \cdots, v_n\}$ = an orthonormal basis of V_1
$\{w_1, \cdots, w_m\}$ = an orthonormal basis of V_2
$\{v_1^*, \cdots, v_n^*\}$ = the dual basis of $\{v_1, \cdots, v_n\}$.

Take $h \in L(V_1, V_2)$ and its matrix

$$\begin{pmatrix} c_{11} & \cdots\cdots\cdots & c_{1n} \\ \cdots & \cdots\cdots\cdots & \cdots \\ \cdots & \cdots\cdots\cdots & \cdots \\ c_{m1} & \cdots\cdots\cdots & c_{mn} \end{pmatrix} \quad (m \times n\text{-matrix})$$

Assume that

$\Phi(h) = v^* \otimes w \in V_1^* \otimes V_2$, and
$v^* = a_1 v_1^* + \cdots\cdots + a_n v_n^*$, $w = b_1 w_1 + \cdots\cdots + b_m w_m$.

Then we have

$$h(v_1) = \Psi(v^* \otimes w)(v_1) = v^*(v_1)w. \tag{4.1}$$

That is

$$c_{11} w_1 + \cdots\cdots + c_{m1} w_m = a_1(b_1 w_1 + \cdots\cdots + b_m w_m).$$

In consequence, we have n equations

$$\left.\begin{aligned} c_{11} w_1 + \cdots\cdots + c_{m1} w_m &= a_1(b_1 w_1 + \cdots\cdots + b_m w_m) \\ c_{12} w_1 + \cdots\cdots + c_{m2} w_m &= a_2(b_1 w_1 + \cdots\cdots + b_m w_m) \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ c_{1n} w_1 + \cdots\cdots + c_{mn} w_m &= a_n(b_1 w_1 + \cdots\cdots + b_m w_m) \end{aligned}\right\} \tag{4.2}$$

By (4.2) we have the following:

$$
\begin{array}{lll}
c_{11}=a_1 b_1 & c_{21}=a_1 b_2 \cdots\cdots\cdots c_{m1}=a_1 b_m \\
c_{12}=a_2 b_1 & c_{22}=a_2 b_2 \cdots\cdots\cdots c_{m2}=a_2 b_m \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
c_{1n}=a_n b_1 & c_{2n}=a_n b_2 \cdots\cdots\cdots c_{mn}=a_n b_m
\end{array}
$$

Therefore, by definition of tensor we have the follwoing equation;

$$\Phi(h)=v^*\otimes w=\sum_{\substack{i=1,\cdots,m \\ j=1,\cdots,n}} c_{ij}\, v_j^*\otimes w_i$$

It also follows from (4.1) that Φ is a linear map.
Furthermore,

$$\Phi\circ\Psi=1_{V_1^*\otimes V_2} \text{ and } \Psi\circ\Phi=1_{L(V_1,\, V_2)}$$

are clear. ■

Recall that for each $\gamma\in\tilde{G}$, (π_γ, V_γ) is an irreducible, finite dimensional, and unitary representation of G (compact Lie group) (see Theorem 3.7).
We difine

$$\chi_\gamma : G \longrightarrow C$$

by

$$\chi_\gamma(g)=\mathrm{tr.}\,\pi_\gamma(g) \text{ (trace of } \pi_\gamma(g)).$$

In this case, χ_γ is called the *character* of (π_γ, V_γ).
We shall use only Haar integrals in this section.

Theorem 4.2. Under the above circumstances, let $\{v_1, \cdots\cdots, v_n\}$ be an orthonormal basis of V_γ and $\{v_1^*, \cdots\cdots, v_n^*\}$ be the dual basis of $\{v_1, \cdots\cdots, v_n\}$. Then the following holds.

(i) $\sum_{i,j=1}^{n} v_i^*(\pi_\gamma(g)v_j)\, \overline{v_i^*(\pi_\gamma(g)v_j)}=n$

for all $g\in G$.

(ii) For v, $w\in V_\gamma$ and v^*, $w^*\in V_\gamma^*$

$$\int_G v^*(\pi_\gamma(g)v)\, \overline{w^*(\pi_\gamma(g)w)}\, dg=\frac{1}{n}\langle v, w\rangle \langle v^*, w^*\rangle.$$

(iii) $\int_G \chi_\gamma(g)\, \overline{\chi_\gamma(g)}\, dg=1.$

(iv) In the isomorphism

$$\Psi : V_\gamma^* \otimes V_\gamma \longrightarrow L(V_\gamma, V_\gamma)$$

we have

$$\Psi(\chi_\gamma) = I.$$

Proof. (1). We put

$$\pi_\gamma(g) = \begin{pmatrix} a_{11} \cdots\cdots\cdots a_{1n} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{n1} \cdots\cdots\cdots a_{nn} \end{pmatrix}.$$

Since $\pi_\gamma(g)$ is unitary, *i. e.*,

$$\pi_\gamma(g)\ \overline{{}^t\pi_\gamma(g)} = I,$$

we have

$$\sum_{i=1}^{n} a_{ij}\ \overline{a_{ik}} = \delta_{jk},$$

and thus

$$\sum_{i,j=1}^{n} |a_{ij}|^2 = |a_{11}|^2 + \cdots\cdots + |a_{1n}|^2 + \cdots\cdots + |a_{nn}|^2 = n.$$

Since

$$v_i^*(\pi_\gamma(g)v_j)\ \overline{v_i^*(\pi_\gamma(g)v_j)} = |a_{ij}|^2,$$

our assertion is true.

(ii) Let

$$P_\gamma : L^2(G) \longrightarrow V_\gamma^* \otimes V_\gamma \qquad (\gamma \in \tilde{G})$$

be the orthogonal projection ($\sum_{\gamma\in\tilde{G}} V_\gamma^* \otimes V_\gamma = L^2(G)$, see Theorem 3.7).
Then the usual action τ (see (3.4)) of $G \times G$ on $L^2(G)$ corresponds to the action $\pi_\gamma^* \hat{\otimes} \pi_\gamma$ (see Definition 3.5) of $G \times G$ on $V_\gamma^* \otimes V_\gamma$. In fact, for $P_\gamma f = v^* \otimes v \in V_\gamma^* \otimes V_\gamma$ and $x, y, z \in G$

$$\begin{aligned}(\tau(x, y)P_\gamma f)(z) &= P_\gamma f(x^{-1}zy) = (v^* \otimes v)(x^{-1}zy) \\ &= v^*(\pi_\gamma(x)^{-1}\ \pi_\gamma(z)\ \pi_\gamma(y)v)\end{aligned}$$

by (3.4) and (3.5). On the other hand,

$$\begin{aligned}((\pi_\gamma^*(x) \otimes \pi_\gamma(y))\ (v^* \otimes v))\ (z) &= (\pi_\gamma^*(x)\ v^* \otimes \pi_\gamma(y)v)\ (z) \\ = \pi_\gamma^*(x) v^*(\pi_\gamma(z)\ \pi_\gamma(y)\ v) &= v^*(\pi_\gamma(x)^{-1}\pi_\gamma(z)\ \pi_\gamma(y)v)\end{aligned}$$

by (3.2) and (3.5).

Since τ is unitary the action $\pi_\gamma^*\hat{\otimes}\pi_\gamma$ on $V_\gamma^*\otimes V_\gamma$ is also unitary. Moreover, since (π_γ, V_γ) is an irreducible, finite dimensional and unitary representation, so is $\pi_\gamma^*\hat{\otimes}\pi_\gamma$. We put

$$h(v^*\otimes v,\ w^*\otimes w)=\int_G v^*(\pi_\gamma(g)v)\ \overline{w^*(\pi_\gamma(g)w)}dg\ .$$

Then it is easy to prove that h is an invariant Hermitian form on $V_\gamma^*\otimes V_\gamma$. Therefore, by Lemma 3.6 there exists a real number c such that

$$h(v^*\otimes v,\ w^*\otimes w)=c\langle v,\ w\rangle\ \langle v^*,\ w^*\rangle$$

(see (3.6)). By (i) and (2.2) we have

$$\int_G \sum_{i,j} v_i^*(\pi_\gamma(g)v_j)\ \overline{v_i^*(\pi_\gamma(g)v_j)}\,dg=n.$$

On the other hand,

$$\int_G \sum_{i,j} v_i^*(\pi_\gamma(g)v_j)\ \overline{v_i^*(\pi_\gamma(g)v_j)}\,dg=\sum_{i,j} c\langle v_j,\ v_j\rangle\ \langle v_i^*,\ v_i^*\rangle=cn^2.$$

So we get $n=cn^2$, and hence $c=\dfrac{1}{n}$.

(iii) By the conclusion of (ii)

$$\int_G v_i^*(\pi_\gamma(g)v_j)\ \overline{v_k^*(\pi_\gamma(g)v_l)}\,dg=\frac{1}{n}\,\delta_{jl}\cdot\delta_{ik} \tag{4.3}$$

Since for $\pi_\gamma(g)=(a_{ij})$, $\chi_\gamma(g)=\sum_{i=1}^{n} a_{ii}$

$$\int_G \chi_\gamma(g)\ \overline{\chi_\gamma(g)}\,dg=\int_G (\sum_i v_i^*(\pi_\gamma(g)v_i))\ \overline{(\sum_j v_j^*(\pi_\gamma(g)v_j))}dg$$

Also using the result of (4.3) and the following fact

$$\int_G v_i^*(\pi_\gamma(g)v_i)\overline{v_j^*(\pi_\gamma(g)v_j)}\,dg=\frac{1}{n}\delta_{ij}$$

we have

$$\int_G \chi_\gamma(g)\overline{\chi_\gamma(g)}\ dg=\sum_{i=1}^{n}\int_G v_i^*(\pi_\gamma(g)v_i)\ \overline{v_i^*(\pi_\gamma(g)v_i)}\,dg=1.$$

(iv) We first note that for $\pi_\gamma(g)=(a_{ij})$

$$\chi_\gamma(g)=\sum_{i=1}^{n} a_{ii}=\sum_{i=1}^{n} v_i^*(\pi_\gamma\,(g)v_i)=\sum_{i=1}^{n}(v_i^*\otimes v_i)(g).$$

Therefore, if we regard χ_γ as an element of $(V_\gamma^*\otimes V_\gamma)$ $(\chi_\gamma\in L^2(G))$ then we can denote

χ_γ by

$$\chi_\gamma = \sum_{i=1}^{n} v_i^* \otimes v_i \tag{4.4}$$

Hence

$$\Psi(\chi_\gamma) = \Psi(\sum_{i=1}^{n} v_i^* \otimes v_i) = \sum_{i=1}^{n} \Psi(v_i^* \otimes v_i).$$

Moreover, for every $v_i \in V_\gamma$

$$\sum_{i=1}^{n} \Psi(v_i^* \otimes v_i)(v_i) = v_i^*(v_i) v_i = v_i$$

and hence we have

$$\Psi(\chi_\gamma) = I. \quad \blacksquare$$

As in the Theorem 3.7 we have

$$L^2(G) = \sum_{\gamma \in \tilde{G}} V_\gamma \otimes V_\gamma^* \quad \text{(a unitary direct sum)}$$

For this expression, let

$$P_\gamma : L^2(G) \longrightarrow V_\gamma \otimes V_\gamma^*$$

be the orthogonal projection. Note that for $f \in L^2(G)$ if we put

$$P_\gamma f = v \otimes v^* \in V_\gamma \otimes V_\gamma^*$$

then for each $g \in G$

$$P_\gamma f(g) = (v \otimes v^*)(g) = v^*(\pi_\gamma(g)^{-1} v).$$

Since, for $\gamma \neq \mu \in \tilde{G}$, $V_\gamma \otimes V_\gamma^*$ and $V_\mu \otimes V_\mu^*$ are orthogonal, we have

$$\int_G \overline{P_\gamma f(g)}\, P_\mu f(g) dg = 0 \quad \text{(see (4.3))}$$

by (ii) of Theorem 4.2. In fact, if we put

$$P_\gamma f = v_\gamma \otimes v_\gamma^* \in V_\gamma \otimes V_\gamma^*$$
$$P_\mu f = w_\mu \otimes w_\mu^* \in V_\mu \otimes V_\mu^*$$

then

$$\int_G \overline{P_\gamma f(g)}\, P_\mu f(g)\, dg$$
$$= \int_G \overline{v_\gamma^*(\pi_\gamma(g^{-1}) v_\gamma)}\; w_\mu^*(\pi_\mu(g^{-1}) w_\mu)\, dg$$
$$= \int_G \overline{v_{\gamma+\mu}^*(\pi_{\gamma+\mu}(g)\, v_{\gamma+\mu})}\; w_{\gamma+\mu}^*(\pi_{\gamma+\mu}(g)\, w_{\gamma+\mu})\, dg$$

$$= \frac{1}{n+m} \langle v_{\gamma+\mu}, w_{\gamma+\mu} \rangle \langle v_{\gamma+\mu}^*, w_{\gamma+\mu}^* \rangle$$

$$= 0,$$

where $\dim_C V_\gamma = n,\ \dim_C V_\mu = m,$

$$v_{\gamma+\mu}^* = (v_\gamma^*, 0) \in V_\gamma^* \oplus V_\mu^*$$

$$w_{\gamma+\mu}^* = (0, w_\mu^*) \in V_\gamma^* \oplus V_\mu^*$$

and $$\pi_{\gamma+\mu}(g) = \begin{pmatrix} \pi_\gamma(g) & 0 \\ 0 & \pi_\mu(g) \end{pmatrix} \quad ((n+m)\times(n+m)-\text{matrix}).$$

We have the following lemma by using the above notations.

Proposition 4.3. For every $f \in L^2(G)$ and $x \in G$

$$P_\gamma f(x) = (\dim V_\gamma) \int_G \overline{\chi_\gamma(g)}\, f(g^{-1}x)\, dg.$$

Proof. By (4.3) it suffices to prove that

$$P_\gamma f(x) = (\dim V_\gamma) \int_G \overline{\chi_\gamma(g)}\, P_\gamma f(g^{-1}x)\, dg.$$

We put $\dim V_\gamma = n$, then by (iv) of Theorem 4.2

$$\chi_r(g) = \sum_{i=1}^{n} v_i^*(\pi_\gamma(g) v_i),$$

where $\{v_1, \cdots\cdots, v_n\}$ is an orthonomal basis of V_γ and $\{v_1^*, \cdots\cdots, v_n^*\}$ is the dual basis of $\{v_1, \cdots\cdots, v_n\}$. Suppose that

$$P_\gamma f(x) = v_i^*(\pi_\gamma(x)^{-1} v_j).$$

Then we have

$$P_\gamma f(g^{-1}x) = \sum_{k=1}^{n} v_i^*(\pi_\gamma(x)^{-1} v_k)\, v_k^*(\pi_\gamma(g) v_j)$$

and thus

$$(\dim V_\gamma) \int_G \overline{\chi_\gamma(g)}\, P_\gamma f(g^{-1}x)\, dg$$

$$= \sum_{k,l} v_i^*(\pi_\gamma(x)^{-1} v_k) \cdot (\dim V_\gamma) \int_G \overline{v_l^*(\pi_\gamma(g) v_l)}\, v_k^*(\pi_\gamma(g) v_j)\, dg$$

$$= \sum_{k,l} v_i^*(\pi_\gamma(x)^{-1} v_k)\, \delta_{kl}\, \delta_{lj} \quad \text{(by (ii) of Theorem 4.2)}$$

$$= P_\gamma f(x). \quad \blacksquare$$

Theorem 4.4. If $f \in L^2(G)$ satisfies the following conditions

(i) $\int_G f(xgyg^{-1})dg = f(x)f(y) \quad (x, y \in G)$

(ii) $P_\tau f \not\equiv 0$

then

$$P_\tau f = (\dim V_\tau)^{-1}\chi_\tau.$$

Proof. In our condition (i) we put $x=e=y$ (e is the identity of G). Then we have

$$P_\tau f(e) = (P_\tau f(e))^2$$

by using (2.2), and hence $P_\tau f(e)=1$. Since $\chi_\tau(e)=n$ by (4.4), we have the following by Proposition 4.3(note that $n=\dim V_\tau$);

$$\begin{aligned}0 &= P_\tau f(e) - \frac{1}{n}\chi_\tau(e)\\ &= n\int_G \overline{\chi_\tau(g)}\, f(g^{-1})\, dg - \int_G \overline{\chi_\tau(g)}\, \chi_\tau(g^{-1})\, dg\\ &= \int_G \overline{\chi_\tau(g)}\, (nf(g^{-1}) - \chi_\tau(g^{-1}))\, dg\end{aligned}$$

Therefore, by (2.5) we get

$$P_\tau f(x) = \frac{1}{n}\chi_\tau(x) = (\dim V_\tau)^{-1}\chi_\tau(x)$$

for all x in G. That is,

$$P_\tau f = (\dim V_\tau)^{-1}\chi_\tau. \quad \blacksquare$$

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