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THE TANGENT BUNDLE OF SMOOTH HOMOTOPY LENS SPACES

Jin Ho Kwak

1. Introduction

Let us consider a variety

$$V_a=\{z : z\in C^{n+1},\ z_1^{a_1}+\cdots+z_{n+1}^{a_{n+1}}=0\},$$

where $a_1, \ldots, a_{n+1}$ are integers greater than 1, and $n>2$. The origin is the only critial point of V_a. The intersection of V_a with the $2n+1$ dimensional standard sphere S^{2n+1}

$$\Sigma_a=\Sigma(a_1, \ldots, a_{n+1})=V_a\cap S^{2n+1},$$

called a *Brieskorn manifold*, has been studied by many authors. A Brieskorn manifold is called a *Brieskorn sphere* if it is a homotopy sphere. For $a=(a_1, \ldots, a_{n+1})$, let $G(a)$ be the graph defined as follow: $G(a)$ has $n+1$ vertices $a_1, \ldots, a_{n+1}$. Two of them, say a_i, a_j, are joined by an edge if and only if the greatest common divisor (a_i, a_j) of a_i and a_j is greater than 1. Brieskorn [1] proved;

THEOREM. *Let $n>2$. Then $\Sigma_a=\Sigma(a_1, \ldots, a_{n+1})=V_a\cap S^{2n+1}$ is a $2n-1$ dimensional topological sphere if and only if $G(a)$ satisfies one of the following conditions*:

1) *$G(a)$ has at least two isolated points,*
2) *$G(a)$ has one isolated point and at least one connected component K with an odd number of vertices such that $(a_i, a_j)=2$ for $a_i, a_j\in K$ $(i\neq j)$.*

The construction of the Brieskorn manifold was generalized by Hamm [2] and Randell [7]. Let

$$f_i(z_1, \ldots, z_{n+m})=\sum_{j=1}^{n+m}\alpha_{ij}z_j^{a_{ij}},\quad 1\leq i\leq m,$$

be polynomials having only one critical point at the origin, where a_{ij}

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are integers greater than 1, α_{ij} are real numbers, and $n>2$. Set

$$V_i=\{z : z\in C^{n+m},\ f_i(z)=0\},$$
$$V_a=V_1\cap V_2\cap\ldots\cap V_m,$$

and

$$\Sigma_a=V_a\cap S^{2(n+m)-1}.$$

Then, each $V_i-\{0\}$ is a smooth manifold of dimension $2n+2m-2$. This Σ_a is called a *generalized Brieskorn manifolds.* Moreover, it is called a *generalized Brieskorn sphere* if it is a homotopy sphere.

For our purpose, we *assume* that

(I) grad $f_1, \ldots,$ grad f_m are linearly independent on $V_a-\{0\}$,

(II) Σ_a is a homotopy sphere (of dimension $2n-1$).

REMARK. (a) Assumption (I) is true if the a_{ij} are independent of i and the real matrix (α_{ij}) has no zero subdeterminant. Of course, it is easy to choose (α_{ij}) with this property. And $V_a=V_1\cap\ldots\cap V_m$ is a complete intersection with its only singularity at the origin, so that dim $\Sigma_a=2n-1$.

(b) Let a_{ij} be independent of i, say $a_{ij}=a_j$ for all i, and assume that the real matrix (α_{ij}) has no zero subdeterminant. Construct a graph $G(a)$ as before, and let $\#_a$ be the number of the connected components K of $G(a)$ such that K has odd vertices and for two different vertices a_i, a_j of K, $(a_i, a_j)=2$. Then, (by Hamm's theorem [2]) $\#_a\geq m+1$ implies that Σ_a is a homotopy sphere.

For a given prime p and a generalized Brieskorn sphere Σ_a, define a free cyclic Z_p-action on Σ_a as follows: Choose natural numbers b_j so that $a_{ij}b_j=h \pmod p$, $h\neq 0$ for all i, j. Let $T(b_1, \ldots, b_{n+m})$ be a map on C^{n+m} defined by

$$T(b_1,\ \ldots,\ b_{n+m})(z_1,\ \ldots,\ z_{n+m})=(\zeta^{b_1}z_1,\ \ldots,\ \zeta^{b_{n+m}}z_{n+m}),$$

where $\zeta=\exp(2\pi i/p)$. The map $T(b_1, \ldots, b_{n+m})$ generates a cyclic group Z_p of order p, under which the homotopy sphere Σ_a is invariant. The orbit space Σ_a/Z_p will be called a *lens space*, and denoted by $L(p;\ a;\ b)$. The map $T(b_1, \ldots, b_{n+m})$ will be denoted by T when there is no ambiguity. Note that we can assume that $a_{ij}b_j=1 \pmod p$ for all i, j without loss of generality by choosing a suitable generator of Z_p.

2. Main Results

Define a Z_p-action on $\Sigma_a \times C$ by $\bar{T}(z, w) = (T(z), \zeta w)$, where $\zeta = \exp(2\pi i/p)$, and $T(z) = T(b_1, \ldots, b_{n+m})(z)$ as before, so that the natural projection from $\Sigma_a \times C$ to Σ_a is equivariant, that is, it commutes with the Z_p-actions. By taking quotients, one can get the canonical complex line bundle γ over the lens space $L(p; a; b)$. Similarly, one can get $\gamma^c = \gamma \otimes \cdots \otimes r$, ($c$ times) with a Z_p-action on $\Sigma_a \times C$ given by $\bar{T}(z, w) = (T(z), \zeta^c w)$. It can be proved easily that

$$\gamma^{b_1} + \cdots + \gamma^{b_{n+m}} = \Sigma_a \times C^{n+m} / T \times T,$$

where

$$(T \times T)(z, (w_1, \ldots, w_{n+m})) = (T(z), \zeta^{b_1} w_1, \ldots, \zeta^{b_{n+m}} w_{n+m}).$$

First, we describe the tangent bundle of a lens space.

THEOREM 1. *Let $\tau = \tau(L(p; a; b))$ and ε denote the tangent bundle and the trivial one-dimensional real bundle over $L(p; a; b)$ respectively. Then $\tau + \varepsilon + m(\mathrm{re}(\gamma))$ is isomorphic to $\mathrm{re}(r^{b_1} + \cdots + r^{b_{m+n}})$ over $L(p; a; b)$.*

Proof. Recall that

$$f_i(z_1, \ldots, z_{n+m}) = \sum_{j=1}^{n+m} \alpha_{ij} z_j^{a_{ij}},$$
$$V_i = f_i^{-1}(0), \quad 1 \leq i \leq m,$$
$$V_a = V_1 \cap \cdots \cap V_m,$$

and

$$\Sigma_a = V_a \cap S^{2(n+m)-1} \text{ (transversely)}.$$

Let $\tau(\cdot)$ denote the tangent bundle and $\nu(\cdot)$ the normal bundle of the space $(\cdot)$ in C^{n+m}. Then

$$\nu(\Sigma_a) = \nu(S^{2(n+m)-1}) + \nu(V_a)$$

by transversality, and

$$\nu(V_a) = \nu(V_1) + \cdots + \nu(V_m).$$

Hence, the trivial bundle $\Sigma_a \times C^{n+m}$ is isomorphic to

$$\begin{aligned} &\tau(\Sigma_a) + \nu(\Sigma_a) \\ &= \tau(\Sigma_a) + \nu(S^{2(n+m)-1}) + \nu(V_a) \\ &= \tau(\Sigma_a) + \nu(S^{2(n+m)-1}) + \nu(V_1) + \cdots + \nu(V_m) \end{aligned}$$

over Σ_a. But $\nu(S^{2(n+m)-1})$ is clearly isomorphic to the trivial bundle ε over Σ_a. To see $\nu(V_i)'$s over Σ_a, let $f: C^{n+m} \longrightarrow C$ be a polynomial defined by

$$f(z_1, \ldots, z_{n+m}) = \sum_{j=1}^{n+m} \alpha_j z_j^{a_j},$$

(as one of f_i's), and let $V=f^{-1}(0)$. Then

$$\text{grad } f(z) = (\alpha_1 a_1 \bar{z}_1^{a_1-1}, \ldots, \alpha_{n+m} a_{n+m} \bar{z}_{n+m}^{a_{n+m}-1})$$

is a cross section of $\nu(V)$, and $\nu(V)=C\cdot\text{grad } f$. Now, with the assumption that $a_j b_j = 1 \pmod{p}$ for all j, we can easily obtain that $T(\text{grad } f(z)) = \zeta\cdot\text{grad } f(Tz)$. Hence, for each $i=1, \ldots, m$, grad f_i is a cross section of $\nu(V_i)$, $\nu(V_i)=C\cdot\text{grad } f_i$, and $T(\text{grad } f_i(z)) = \zeta$ grad $f_i(Tz)$ over Σ_a. Define

$$\phi : \tau(\Sigma_a) + R + C^m \longrightarrow \Sigma_a \times C^{n+m}$$

by

$$\phi(v_z, r, w_1, \ldots, w_m) = (z,\ v + rz + w_1 \text{ grad } f_1(z) + \cdots + w_m \text{ grad } f_m(z)),$$

where v_z is a tangent vector at z, and R, C^m represents the trivial bundle $R\times\Sigma_a$, $C^m\times\Sigma_a$ respectively. Then ϕ is an equivariant isomorphism with an action $dT+I+(\cdot\zeta)$ on $\tau(\Sigma_a)+R+C^m$. Therefore, by taking quotients, it is proved.

Recall that the standard lens space $L^{2n-1}(p)$ is defined as the orbit space of S^{2n-1} by the linear action $T(1, \ldots, 1)$. Since the principal Z_p-bundles

$$S^{2n-1} \longrightarrow L^{2n-1}(p) \text{ and } \Sigma_a \longrightarrow L^{2n-1}(p;a;b)$$

are $(2n-1)$-universal, there are maps

$$f : L^{2n-1}(p) \longrightarrow L^{2n-1}(p;a;b),$$
$$g : L^{2n-1}(p;a;b) \longrightarrow L^{2n-1}(p)$$

such that the induced bundles

$$f^*\gamma = \gamma \quad \text{and} \quad g^*\gamma = \gamma,$$

where γ is the canonical bundle over the suitable lens space. Hence, we have

COROLLARY. *The lens space $L(p;a;b)$ is stably parallelizable, i.e., its tangent bundle is stably trivial, if and only if m (re(γ)) is stably isomorphic to*

$$\mathrm{re}(\gamma^{b_1})+\mathrm{re}(\gamma^{b_2})+\cdots+\mathrm{re}(\gamma^{b_{n+m}})$$

over the standard lens space $L^{2n-1}(p)$, *where* γ *represents the canonical line bundle over* $L^{2n-1}(p)$.

Let v be a prefered generator for $H^2(L(p;a;b);Z)$ and u its mod 2 reduction. Then, we obtain the following total Pontrjagin and Stiefel-Whitney classes of a lens space:

THEOREM 2. 1) $P(L(p;a;b))=(1+v^2)^{-m}\prod_{i=1}^{n+m}(1+b_i^2v^2)$.

2) $w(L(p;a;b))=(1+u)^{-m}\prod_{i=1}^{n+m}(1+b_iu)$.

COROLLARY. *If the lens space* $L(p;a;b)$ *is stably parallelizable, then*

$$(1+v^2)^m=(1+b_1^2v^2)(1+b_2^2v^2)\cdots(1+b_{n+m}^2v^2)$$

in $\sum H^{even}(L^{2n-1}(p);Z_p)\simeq Z_p[v]/(v^n)$.

REMARK. Let $L(p;a;b)$ be defined as an orbit space of a Brieskorn sphere so that $m=1$. Then, we have in Theorem 1

$$\tau+\varepsilon+m(\mathrm{re}(\gamma))\simeq\mathrm{re}(\gamma^{b_1}+\ldots+\gamma^{b_{n+1}}).$$

This is a correction of Orlik's Theorem 3([6], p. 252). And Theorem 2 is a correction of Orlik's Theorem 4 of the same paper. Note that $L(p;a;b)$ is a submanifold of codimension 2 of the classical lens space $L^{2n+1}(p:b_1,\ldots,b_{n+1})$. It is well-known that

$$\tau(L^{2n+1}(p;b_1,\ldots,b_{n+1}))+\varepsilon$$

is isomorphic to

$$\mathrm{re}(\gamma^{b_1}+\gamma^{b_2}+\cdots+\gamma^{b_{n+1}}).$$

Hence, by Theorem 1, the normal bundle of $L(p;a;b)$ in $L^{2n+1}(p;b_1,\ldots,b_{n+1})$ is stably isomorphc to $\mathrm{re}(\gamma)$, whicih is *not* trivial. This gives a correction of an Orlik's computation of the normal bundle ([6], p. 252).

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Kyungpook University
Taegu 635, Korea