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STABILITY IN A CONVEX PROGRAM WITH LINEAR CONSTRAINTS

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1. Introduction

Consider the following ordinary convex program with linear constraints:

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{Subject to} \quad & b-Ax\in Q^* \\ & x\in P, \end{aligned} \tag{1}$$

where f is a proper convex function on R^n, $b\in R^m$, A is an $m\times n$ matrix, P and Q are nonempty convex polyhedral cones in R^n and R^m, respectively, and $Q^*=\{u^*\in R^m \mid \langle u^*, u\rangle \leq 0, \text{ for each } u\in Q\}$ is the polar cone of Q.

Williams [7] has studied the stability of (1) (where f is lanear) as the system parameters are perturbed along a certain direction. In [3], Robinson extended Williams' result by considering only the magnitude of the perturbation, not its direction. In that paper, he studied the stability of linear programs and also provided bounds for the distance from a solution of an original program to the solution sets of the perturbed linear system. For general mathematical programs, the stability of optimal solution sets has been investigated by Evans and Gould [1]; their results were subsequently extended by Greenberg and Pierskalla [2]. But, those results only dealt with qualitative bounds for changes in the optimal solution set. In this paper, the quantitative bounds for the changes in the optimal solution set are obtained.

In section 2, the bifunction associated with a convex program (1) is used to define perturbed convex programs of (1). This is following by some discussions on the relationship between the adjoint bifunction and duality. Regularity conditions on (1) then are given. It is shown that these regularity conditions are necessary and sufficient for the stability of the convex program (1). In section 3, some results from Robinson's

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paper [5] are used to obtain bounds for the distance from solutions of (1) and its dual program to the solution sets of the perturbed program and its associated dual program.

2. Duality and regularity conditions

Associated with an ordinary convex program (1), there is a bifunction F from R^m to R^n such that for each $u \in R^m$, Fu is a function on R^n defined by

$$Fu = f + \delta(\cdot \mid S(u)), \qquad (2)$$

where $S(u) = \{x \mid (b+u) - Ax \in Q^*,\ x \in P\}$ is a perturbed constraint system of (1), and $\delta(\cdot \mid S(u))$ is the indicator function of set $S(u)$ [6]. The convex program generated by a perturbation u can be defined by

$$(\text{Inf } F)(u) = \text{Inf}(Fu) = \text{Inf}_x (Fu)(x),$$

where Inf F is a function from R^m to R and is called the perturbation of F [6]. The ordinary convex program (1) then can be represented by $\text{Inf}_x (F0)(x)$. Therefore, instead of studying an ordinary convex program (1) and its perturbation, we could just as well study it in terms of the associated bifunction F and its perturbation function Inf F We shall only mention some important results concerning a bifunction F in this paper. See [6, Chapter 28] for details.

For any convex bifunction F from R^m to R^n, its adjoint F^*, is defined as a bifunction from R^n to R^m given by

$$(F^* x^*)(u^*) = \text{Inf}_{x,u} \{(Fu)(x) - \langle x, x^* \rangle - \langle u, u^* \rangle\}.$$

Note that this definition is equivalent to the one in [6] in which perturbation is given at the right hand side of the constraint system whereas here, it is given at the left hand side.

The adjoint of F defined in (2) then can be calculated as follows:

$$\begin{aligned}(F^* x^*)(u^*) &= \text{Inf}_{x,u} \{(Fu)(x) - \langle x, x^* \rangle - \langle u, u^* \rangle\} \\ &= \text{Inf}_{x,u} \{f(x) + \delta(x \mid (b+u) - Ax \in Q^*, x \in P) - \langle x, x^* \rangle - \langle u, u^* \rangle\} \\ &= \text{Inf}_{x \in P, v \in Q^*} \{f(x) - \langle x, x^* \rangle + \langle b - Ax, u^* \rangle - \langle v, u^* \rangle\} \\ &= \text{Inf}_{x \in P} \{f(x) + \langle b - Ax, u^* \rangle - \langle x, x^* \rangle\} + \text{Inf}_{v \in Q^*} \{-\langle v, u^* \rangle\} \\ &= \text{Inf}_{x \in P} \{f(x) + \langle b - Ax, u^* \rangle - \langle x, x^* \rangle\}, \text{ if } u^* \in Q.\end{aligned}$$

The function $\operatorname{Sup} F^*$ defined by $(\operatorname{Sup} F^*)(x^*) = \operatorname{Sup}_{u^*}(F^*x^*)(u^*)$ is then called the perturbation function of F^*. Therefore, the dual program of (1) is $(\operatorname{Sup}_{u^*} F^*)(0) = \operatorname{Sup}_{u^*}(F^*0)(u^*)$, where

$$((F^*0)(u^*) = \begin{cases} \operatorname{Inf}_{x\in P} f(x) + \langle b - Ax, u^* \rangle, & \text{if } u^* \in Q \\ -\infty & , \text{ if } u^* \notin Q. \end{cases} \tag{3}$$

A vector $\bar{u}^* \in Q$ is called a Kuhn-Tucker vector of (1) if $(F^*0)(\bar{u}^*)$ is finite and equal to the optimal value of (1), i.e., $\operatorname{Sup}_{u^*}(F^*0)(u^*) = (F^*0)(u^*) = \operatorname{Inf}_x (F0)(x)$.

If we let $\partial(\operatorname{Inf} F)(u) = \{u^* \mid (\operatorname{Inf} F)(v) \geq (\operatorname{Inf} F)(u) + \langle u^*, v-u \rangle,$ for all $v \in R^m\}$ be the set of all subgradients of $\operatorname{Inf} F$ at u, then it is well known [6, Thm. 29.1] that when the optimal value of (1) is finite, u^* is a Kuhn-Tucker vector of (1) if and only if $-u^* \in \partial$ (Inf F) (0). It is also known that if $\operatorname{Inf} F$ is a proper convex function. then $\partial(\operatorname{Inf} F)(u)$ is nonempty and bounded if and only if $u \in \operatorname{Inf}(\operatorname{dom}(\operatorname{Inf} F)) = \operatorname{Int}\{v \mid (\operatorname{Inf} F)\ (v)$ is finite$\}$ [6, Theorem 23.4].

We now return to the consideration of the stability of convex program (1). We shall prove that the behavior of optimal solution sets of (1) and its dual program (3) under perturbation is closely related to the following regularity conditions. We say (1) is regular if $0 \in \operatorname{Int}$ (dom (Inf F)) and $0 \in \operatorname{Int}$ (dom (Sup F^*)).

It will be shown in the following theorem that the above regularity conditions are necessary and sufficient for the stability of (1).

THEOREM 1. *Assume the bifunction F associated with* (1) *is closed. Then the following are equivalent.*

(a) $0 \in \operatorname{Int}$ (dom(Inf$F$)), $0 \in \operatorname{Int}$ (dom (Sup F^*)).

(b) *The sets of optimal solutions of* (Inf F) (0) *and* (Sup F^*) (0) *are non-empty and bounded.*

(c) *There exists $\varepsilon_0 > 0$ such that for any u with $\|u\| < \varepsilon_0$,* (Inf F)(u) *and its dual are solvable, and for any x^* with $\|x^*\| < \varepsilon_0$,* (Sup F^*) (x^*) *and its dual are solvable, where $\|\cdot\|$ is any l_p norm on R^m and R^n.*

Proof. We shall prove (a) is equivalent to (b) first.

If not both (Inf F) (0) and (Sup F^*)(0) are solvable, then at least one of them is inconsistent, thus, (a) and (b) both are false.

If both (Inf F) (0) and (Sup F^*) (0) are solvable, then $0 \in \text{Int}$ (dom (Inf F)) $\Longleftrightarrow \partial$(Inf F) (0) is nonempty and bounded. [6, Thm. 23. 4]

$\Longleftrightarrow -\partial$(Inf F) (0) is nonempty and bounded.

$\Longleftrightarrow$ The set of all Kuhn–Tucker vectors for (Inf F) (0) is nonempty and bounded. [6, Thm. 29. 1]

$\Longleftrightarrow$ The set of optimal solutions of (Sup F^*) (0) is nonempty and bounded. [6, Thm. 30. 4, 30. 5]

By the same arguments, we can prove $0 \in \text{Int}$ (dom(Sup F^*)) if and only if the solution set of (Inf F) (0) is nonempty and bounded.

Now, to prove (a) is equivalent to (c), Note that

$$(\text{Inf } F)(u) = \operatorname*{Inf}_x (F\ u)(x) = \operatorname*{Inf}_x \{f(x) + \delta(x \mid S(u))),$$

for any $u \in \text{dom}(\text{Inf } F)$.

Fix a given $u \in \text{dom}(\text{inf } F)$, define a new bifunction $\bar{F}$ by

$$(\text{Inf } \bar{F})(w) = \operatorname*{Inf}_x (\bar{F}\ w)(x) = \text{Inf}\{f(x) + \delta(x \mid S(u+w))\}.$$

Then we have (Inf $\bar{F}$)(0) = (Inf F)(u), and the dual program of (Inf F)(u) is (Sup $\bar{F}^*$)(0), where $\bar{F}^*$ is the adjoint of $\bar{F}$.

Assume (a) is true. Then $0 \in \text{Int}$ (dom(Inf F)) implies there exists a neighborhood N_1 of $0 \in R^m$ with radius ε' such that $N_1 \subset \text{Int}$ (dom (Inf F)) and $(\operatorname*{Inf}_x F)(u)$ is finite for all $u \in N$. Let $\varepsilon_1 = \frac{\varepsilon'}{4}$, then for any $u_1$with $\|u\| < \varepsilon_1$, the set $N_2 = \{v-u \mid \|v-u\| < \varepsilon_1\} \subset N_1$ is a neighborhood of 0 with radius ε_1. Furthermore, (Inf $\bar{F}$)(w) = Inf($F(u+w)$ is finite for all $w \in N_2$. Therefore, $0 \in \text{Int}(\text{dom}(\bar{F}))$ which implies Inf $\bar{F}(0) = \text{Sup } \bar{F}^*(0)$ [6, Thm. 30. 3]. But, Inf $\bar{F}(0)$ = (Inf F) (u) and Sup $\bar{F}^*(0)$ is the dual of (Inf F)(u), we have (Inf F)(u) and its dual are solvable.

Dually, there exists a neighborhood, N_3, of $0 \in R^n$ with radius ε_2 such that for all $x^* \in N_3$, (Sup F^*) (x^*) and its dual are solvable. Let $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$, then ε_0 is the number required by (c). (c) implies (a) is obvious. Q. E. D.

The regularity conditions given here are stronger than the normality condition given by Rockafellar [6]. However, our regularity conditions are equivalent to the stability of a convex program (1) in the sense stated in (c) of Theorem 1.

3. Error Bounds

It seems reasonable to say that Inf F is stable at 0 if each program Inf(Fu) and its dual program are solvable for all u in some neighborhood of 0, and if, in addition, the optimum solution sets of Inf (Fu) and its dual program do not jump away from any optimal solutions of Inf$(F0)$ and Sup (F^*0), We should like to obtain an error bound for the distance from any optimal solutions of Inf $(F0)$ and Sup(F^*0) to the optimal solution sets of Inf (Fu) and its dual program.

We shall assume that the regularity conditions hold for the convex program (1) throughout this section.

Recall that feasible solutions of the dual program of (1) can be described as vectors $u^*\in Q$ such that the infimum of the proper convex function $f(x)+\langle b-Ax, u^*\rangle$ on R^n is finite. If f is continuously differentiable on R^n, then feasible solutions of the dual program of (1) are those $u^*\in Q$ such that there is $x\in P$ with

$$A^*u^*-\nabla f(x)\in P^*,$$

where A^* is the adjoint of A. For any such u^* and x one has

$$(F^*0)(u^*)=f(x)+\langle b-Ax, u^*\rangle.$$

Therefore, the dual program of (1) has an equivalent form:

$$\begin{aligned} &\text{Maximize } f(x)+\langle b-Ax, u^*\rangle \\ &\text{subject to } A^*u^*-\nabla f(x)\in P^* \\ &u^*\in Q. \end{aligned} \tag{4}$$

The Kuhn-Tucker conditions on the convex program (1) then determine the following nonlinear system:

$$b-Ax\in Q^* \tag{5}$$
$$A^*u^*-\nabla f(x)\in P^* \tag{6}$$
$$\langle b-Ax, u^*\rangle=0 \tag{7}$$
$$\langle A^*u^*-\nabla f(x), x\rangle=0 \tag{8}$$
$$x\in P \tag{9}$$
$$u^*\in Q \tag{10}$$

The conditions (5), (6), (9), and (10) are the constraints for the feasibility of (1) and its dual, the complementary conditions (7) and (8) guarantee that there is no duality gap.

Note that if the convex program (1) is perturbed through its associated perturbation function by a vector w, only (5) and (7) will be affected in the Kuhn–Tucker conditions of the perturbed system.

In [4], Robinson defined that a linear system, $d-Dx\in K$, $x\in C$ is regular if $d\in \text{int}\{D(C)+K\}$. He then extended this concept to define a nonlinear system is regular [5] if its linearized system is regular.

Denote the solution set of the perturbed system of (5)$\sim$(10). Then $\sum(0)$ is the set of all Kuhn–Tucker vectors of the unperturbed convex program. By Theorem 1, the regularity conditions imply $0\in \text{Int}\{w|\sum(w)$ is nonempty and bounded$\}$. Since a convex system is contained in its linearized system, our regularity conditions imply the system (5)$\sim$(10) is also regular in the sense of Robinson's [5]. We shall restate the Corollary 1 from [5] in our version as follows.

THEOREM 2 [5]. *Assume that* $0\in\text{Int}$ (dom Inf F), $0\in\text{Int}$(dom Sup F^*), *and* $f\in C^2$. *Then there exists a constant* $\alpha>0$ *such that for each* $(x_0, u_0^*)\in\sum(0)$, *there are neighborhood* N_1 *of* 0 ($\in R^m$) *and* N_2 *of* (x_0, u_0^*). *with the property that for each* $w\in N_1$ *and for each* $(x, u^*)\in N_2$, *we have* $\sum(w)$ *is nonempty and bounded, and*

$$d[(x, u^*), \textstyle\sum(w)]\le \alpha d[(b-Ax, A^*u^*-\nabla f(x), \langle b-Ax, u^*\rangle, \langle A^*u^*-\nabla f(x), x\rangle); Q^*\times P^*\times\{0\}\times\{0\}], \tag{11}$$

where $d[y, B]=\inf\{\|y-b\| \; |b\in B\}$ *is the distance from* y *to a set* B.

(11) describes an error bound for the distance from any given point (x, u^*) near (x_0, u_0^*) to the optimal solution sets of the perturbed program and its dual in terms of the amount by which (x, u^*) fails to satisfy the perturbed nonlinear system of (5)$\sim$(10). In particular, we have

$$d[(x_0, u_0^*), \textstyle\sum(w)]\le\alpha d[(b+w-Ax_0, \; A^*u_0^*-\nabla f(x_0), \langle b+w-Ax_0, u_0^*\rangle, \langle A^*u_0^*-\nabla f(x_0), x_0\rangle); Q^*\times P^*\times\{0\}\times\{0\}]. \tag{12}$$

We shall use this result to obtain an error bound for the distance from a pair of optimal solutions of (1) and its dual program to the optimal solution sets of a perturbed convex program and its dual program.

In the remainder of this section, we shall let $\|\cdot\|$ denote a norm defined on R^{m+n+2} such that $\|(u, x, c, d)\|=\|u\|+\|x\|+|c|+|d|$ and let

$$\delta=\text{Sup}\{\|x\| \,|x\in\textstyle\sum(0)\}\to\pi=\text{Sup}\;\{\|u\| \,|(x, y)\in\textstyle\sum(0)\}$$

THEOREM 3. *Under the same assumptions as in* Theorem 2, *there exists*

$\beta>0$ *such that for each* $(x_0, u_0^*) \in \sum(0)$, *there is an* $\varepsilon>0$ *such that for each* w *with* $\|w\|<\varepsilon$, *we have*

(i) $\sum(w)$ *is non-empty and bounded*

(ii) $d[(x_0, u_0^*), \sum(w)] \leq \in\beta$.

Proof. Let $\varepsilon=\min\{\varepsilon_1, \varepsilon_2\}$, where ε_1 and ε_2 are numbers determined in Theorem 1 and Theorem 2, respectively. Then (i) follows by Theorem 1. By Theorem 2 and (12), there exists an $\alpha>0$ such that

$$\begin{aligned} d[(x_0, u_0^*), \textstyle\sum(w)] \leq & \alpha d[(b+w-Ax_0, \ A^*u_0^*-\nabla f(x_0), \\ & \langle b+w-Ax_0, u_0^*\rangle, \ \langle A^*u_0^*-\nabla f(x_0), x_0\rangle); Q^*\times P^* \{0\} \times \{0\}] \\ & =\alpha\{d[b+w-Ax_0, Q^*]+d[A^*u_0^*-\nabla f(x_0), P^*] \\ & +|\langle b+w-Ax_0, u_0^*\rangle|+|\langle A^*u_0^*-\nabla f(x_0), x_0\rangle|\} \end{aligned}$$

for all w with $\|w\|<\varepsilon$.

Note that

$$\begin{aligned} & d|b+w-Ax_0; Q^*] \geq \|(b+w-Ax_0)-(b-Ax_0)\|=\|w\| \leq \varepsilon, \\ & d[A^*u_0^*-\nabla f(x_0); p^*]=0, \ \text{ since } A^*u_0^*-\nabla f(x_0) \in p^*, \\ & |\langle b+w-Ax_0, u^{0*}\rangle| = |\langle b-Ax_0, u_0^*\rangle+\langle w, u_0^*\rangle| \\ & \qquad =\|w\| \ \|u^{0*}\| \\ & \qquad \leq \varepsilon\pi \end{aligned}$$

and

$$|\langle A^*u_0^*-f(x_0), x_0\rangle|=0,$$

we have

$$\begin{aligned} d[(x_0, u_0^*), \textstyle\sum(w)] & \leq \alpha\varepsilon+\alpha\varepsilon\pi \\ & =\varepsilon(1+\pi) \\ & =\varepsilon\beta. \end{aligned}$$

Q. E. D.

Let $P(w)$ and $D(w)$ be optimal solution sets of (Inf Fw) and its dual program, respectively. It is obvious that $P(w)\times D(w)\supset\Sigma(w)$. Another interesting problem is to measure error bounds of the distance from a pair of optimal solutions of a perturbed convex program and its dual program to optimal solutions sets of the unperturbed convex program and its dual, $P(0)$ and $D(0)$. While these bounds can be obtained in linear programming problems [3], some difficulties are involved in nonlinear programming problems and the bounds have not yet been obtained.

Let (x_0, u_0^*) be any point in $\sum(0)$. By theorem 3, there exists $\beta>0$

and ε (depending on (x_0, u_0^*)) such that $d[(x_0, u_0^*), \sum(w)] \leq \varepsilon\beta$ for each w with $\|w\| \leq \varepsilon$. Let N_ε (0) denote a neighborhood of $0 \in \mathbf{R}^m$ with radius ε and let $\varepsilon_1 = \frac{\varepsilon}{8}$. Then for each fixed $w \in N_{\varepsilon_1}$ (0), the set $\{v-w \mid \|v-w\| < 2\varepsilon_1\}$ is a neighborhood of 0 containing w and $-w$, Define $(\text{Inf } \bar{F})(v) = (\text{Inf } F)(w+v)$ as in theorem 1, then $(\text{Inf } \bar{F})(0) = (\text{Inf } F)(w)$ and $(\text{Inf } \bar{F})(-w) = (\text{Inf } F)(0)$. Applying theorem 3 to the perturbation function Inf $\bar{F}$ gives a $\beta > 0$ such that for each $(\bar{x}, \bar{u}^*) \in \sum(0) = \sum(w)$, there exists $\varepsilon_2 > 0$ (depending on $\bar{x}, \bar{u}^*$)) such that

$$d[\bar{x}, \bar{u}^*) \bar{\sum}(v)] \leq \varepsilon_2\beta, \text{ for each } v \text{ with } \|v\| < \varepsilon_2. \tag{13}$$

Unfortunately, there is no guarantee that $\varepsilon_2 \geq \varepsilon$. Therefore $\bar{\sum}(-w) = \sum(0)$ may not be used in (13). However, if this is the case, then we can have the following result.

THEOREM 4. *Under the same assumption as in* theorem 2, *let* $\varepsilon, \varepsilon_1$, *and* w *be defined and chosen as above. If there is* $(\bar{x}, \bar{u}^*) \in \bar{\sum}(w)$ *such that its associated* ε_2 *in* (13) *satisfies* $\varepsilon_2 \geq 2\varepsilon_1$, *then there exists* $\bar{\beta} > 0$ *such that*

$$d[\bar{x}; P(0)] + d[\bar{u}^*; D(0)] \leq \varepsilon_1 \bar{\beta}$$

Proof. $\varepsilon_2 \geq 2\varepsilon_1$ implies $\|-w\| < \varepsilon_2$ Therefore,

$$\begin{aligned} d[(x, u^*); \sum(-w)] &= d|(\bar{x}, \bar{u}^*); \sum(0)] \\ &\leq \gamma d[(b-Ax, A^*\bar{u}^* - \nabla f(\bar{x}), \langle b-A\bar{x}^*, \bar{u}^*\rangle, \\ &\quad \langle A^*\bar{u}^* - \nabla f(\bar{x}), \bar{x}\rangle); Q^* \times P^* \times \{0\} \times \{0\}) \end{aligned}$$

for some $\gamma > 0$.

Note that $d[(\bar{x}, \bar{u}^*); \sum(0)] \geq d[\bar{x}; P(0)] + d[\bar{u}^*; D(0)]$ by the definition of our norm. Let $\bar{x}_0$ and $\bar{u}_0^*$ be points in $P(0)$and in $D(0)$, closest to $\bar{x}$ and $\bar{u}^*$, respectively. Then $\|\bar{x} - \bar{x}_0\| = d[\bar{x}, P(0)]$ and $\|\bar{u}^* - \bar{u}_0^*\| = d(\bar{u}^*; D(0)]$. Now, since

$$\begin{aligned} &d[b-A\bar{x}; Q^*] = \|(b-A\bar{x}) - (b+w-A\bar{x})\| = \|w\| \leq 2\varepsilon_1, \\ &d[A^*\bar{u}^* - \nabla f(\bar{x}), P^*] = 0, \\ &|\langle b-A\bar{x}, \bar{u}^*\rangle| = |\langle b-A\bar{x}, \bar{u}^*\rangle - \langle b+w-A\bar{x}, \bar{u}^*\rangle| \\ &\qquad = |\langle w, \bar{u}^*\rangle| \\ &\qquad = \|w\| \ \|\bar{u}^*\| \\ &\qquad \leq 2\varepsilon_1(\|\bar{u}^* - u_0^*\| + \|\bar{u}_0\|) \\ &\qquad \leq 2\varepsilon_1(d(\bar{u}^*, D(0)| + \pi), \end{aligned}$$

and

$$|\langle A^*\bar{u}^* - \nabla f(x), \bar{x}\rangle| = 0,$$

we have

$$d[x;P(0)] + d(\bar{u}^*, D(0)] \leq 2\gamma\varepsilon_1 d[\bar{u}^*, D(0)] + 2\gamma\varepsilon_1\pi + 2\gamma\varepsilon_1$$

or

$$d[\bar{x};P(0)] + (1-2\gamma\varepsilon_1)d(\bar{u}^*, D(0)) \leq 2\gamma\varepsilon_1(1+\pi)$$

or

$$(1-2\gamma\varepsilon_1)[d[\bar{x};P(0)] + d[\bar{u}^*, D(0)] \leq 2\gamma\varepsilon_1(1+\pi),$$

if we choose $\varepsilon_1 < \frac{1}{2\gamma}$

or

$$d[\bar{x};P(0) + d[\bar{u}^*, D(0)] \leq \frac{2\gamma\varepsilon_1(1+\pi)}{(1-2\gamma\varepsilon_1)} = \varepsilon_1\bar{\beta},$$

where $\bar{\beta} = \frac{2\gamma(1+\pi)}{1-2\gamma\varepsilon_1}$. Q. E. D.

4. Summary

The stability of perturbed ordinary convex programs is studied through the concept of bifunction and its associated perturbation function. It is followed by a detailed discussion about the relationship between adjoint bifunctions and dual programs. Regularity conditions for an ordinary convex program are then given. It is proved that the regularity conditions given here are necessary and sufficient for the stability of an ordinary convex program. An error bound for the distance from a pair of optimal solutions of Inf ($F0$) and Sup (F^*0) to the optimal solution sets of Inf (Fu) and its dual program is given. An error bound for the distance from a pair of optimal solution of Inf (Fu) and Sup ($F^* u$) to the optimal solution sets of Inf($F0$) and its dual program is also discussed.

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