A THEOREM CONNECTING TWO-DIMENSIONAL WEYL OPERATOR AND H-FUNCTION TRANSFORM

R.K. RAINA

1. Preliminaries and Notations

Let Ω denote the class of all such functions $f(x,y)$ which are differentiable partially any number of times and if it and all of its partial derivatives are $0(|x|^{-r}\cdot|y|^{-s})$, for all r and s, as both x and y increase without limit.

Corresponding to the two-dimensional Riemann-Liouville fractional operator due to Al-Bassam [1], we may define the two-dimensional Weyl fractional operator of a function $f(x,y)$ as follows:

$$(1.1)\quad W_x{}^{\alpha}W_y{}^{\beta}f(x,y)=\frac{(-1)^{\alpha+\beta}}{\Gamma(-\alpha)\Gamma(-\beta)}\int_x^{\infty}\int_y^{\infty}(u-x)^{-\alpha-1}(v-y)^{-\beta-1}\cdot f(u,v)\,dudv,$$

for $\alpha<0$, $\beta<0$.

For all other real values of α and β, we define

$$(1.2)\qquad W_x{}^{\alpha}W_y{}^{\beta}f(x,y)=D_{x,y}^{m+n}(W_x^{\alpha-m}W_y^{\beta-n}f(x,y)),$$

where m,n are positive integers such that $m-1\leqslant\alpha<m$, $n-1\leqslant\beta<n$, and $D_{x,y}^{m+n}$ stands for the operator $\partial^{m+n}/\partial x^m\partial y^n$.

The representations (1.1) and (1.2) exist, whenever $f\in\Omega$, see the observations made in the paper of Miller [6, p. 82] for the corresponding single analogue operators (1.1)–(1.2).

It may be noted that when $\alpha=\beta=0$, we have the identity operator

$$(1.3)\qquad W_x{}^0W_y{}^0f(x,y)=f(x,y).$$

This property follows from (1.2) in conjuction with (1.1), since

$$W_x{}^0W_y{}^0f(x,y)=\partial^2/\partial x\partial y\left(\int_x^{\infty}\int_y^{\infty}f(u,v)\,dudv\right)$$
$$=f(x,y),\ \text{because } f(x,y)\in\Omega.$$

Received December 20, 1982, Revised October 25, 1983.

Also, if $\alpha=m$, $\beta=n$, (m, n positive integers), then

$$(1.4)\qquad W_x^m W_y^n f(x,y)=D_{x,y}^{m+n} f(x,y),$$

which follows at once from (1.2) and (1.3).

For convenience sake, we make use of the symbolic forms $(a_i,\alpha_i)_{1,P}$ and $(a_i,\alpha_i,A_i)_{1,P}$, $p\geqslant 1$, to condense the array of P-parameters

$$(1.5)\qquad (a_1,\alpha_1),\ldots,(a_P,\alpha_P) \text{ and } (a_1,\alpha_1,A_1),\ldots,(a_P,\alpha_P,A_P),$$

the array being considered to be empty if $P=0$.

2. Introduction

In a very recent paper Raina and Koul [10] obtained a result for the H-function transform of a function $t^q f(t)$ (q real) in the Weyl fractional calculus [6]. The H-function transform [4, p. 142] is defined by

$$(2.1)\qquad \bar{f}(p)=\int_0^\infty H_{P,Q}^{M,N}\left[(pt)^h\left|\begin{matrix}(a_i,\alpha_i)_{1,P}\\(b_i,\beta_i)_{1,Q}\end{matrix}\right.\right]f(t)\,dt,$$

where M, N, P, Q are integers such that $1\leq M\leq Q$, $0\leq N\leq P$ and $h>0$. The kernel is the well known H-function of C. For [3, p. 408]; see [8] for its definition and other related details.

We deem it convenient to mention the Raina-Koul result in the following form:

THEOREM 1 (Raina and Koul [10, p. 276]). *If $\bar{f}(p)$ given by* (2.1) *is such that $W_q{}^q\bar{f}(p)$ (the single-analogue operator of* (1.1)) *exists. Then, for all real values of q,*

$$(2.2)\qquad (-1)^q\ W_p{}^q\bar{f}(p)=\phi[t^q f(t);p],$$

where

$$(2.3)\qquad \phi[t^q f(t);p]=\int_0^\infty H_{P+1,Q+1}^{M+1,N}\left[(pt)^h\left|\,L\,\right.\right]t^q f(t)\,dt,$$

provided that the integrals are absolutely convergent, and, for convenience, L denotes (in abbreviated form (1.5)) *the parameter sets given by*

$$(2.4)\qquad L=\begin{cases}(a_i-\dfrac{q\alpha_i}{h},\alpha_i)_{1,P},\ (-q,h)\\[2ex](0,h),\ (b_i-\dfrac{q\beta_i}{h},\beta_i)_{1,Q}.\end{cases}$$

This paper develops the extension of the aforementioned Theorem 1 to two dimensions by invoking the two-dimensional operator (1.1). Some special cases are also briefly pointed out.

3. Extension of Theorem 1

We first consider a two-dimensional transform by the integral equation

$$(3.1)\qquad \bar{g}(p,q)=\int_0^\infty\int_0^\infty H^*[(px)^h,(qy)^k]\ g(x,y)\,dx\,dy,\quad h>0,\ k>0.$$

The kernel of the integral equation (3.1) is a *special case* of the H-function of two variables Mittal and Gupta [7, pp. 117–118] (see also [13]) possessing the notational form

$$(3.2)\qquad H^*[x,y]=H^{0,0:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\ y\end{matrix}\middle|\begin{matrix}(a_i;\alpha_i,A_i)_{1,p_1}:(c_i,\gamma_i)_{1,p_2}\\ (b_i;\beta_i,B_i)_{1,q_1}:(d_i,\delta_i)_{1,q_2}\end{matrix}\begin{matrix};(e_i,E_i)_{1,p_3}\\ ;(f_i,F_i)_{1,q_3}\end{matrix}\right],$$

and is essentially defined in terms of a double contour integral of Mellin-Barnes type. If we let

$$(3.3)\qquad \Delta=-\sum_1^{p_1}(\alpha_i)-\sum_1^{q_1}(\beta_i)+\sum_1^{m_2}(\delta_i)-\sum_{m_2+1}^{q_2}(\delta_i)+\sum_1^{n_2}(\gamma_i)-\sum_{n_2+1}^{p_2}(\gamma_i),$$

and

$$(3.4)\qquad \nabla=-\sum_1^{p_1}(A_i)-\sum_1^{q_1}(B_i)+\sum_1^{m_3}(F_i)-\sum_{m_3+1}^{q_3}(F_i)+\sum_1^{n_3}(E_i)-\sum_{n_3+1}^{p_3}(F_i),$$

then it is well known that for $\Delta>0$ and $\nabla>0$, the function $H^*[x,y]$ converges absolutely and defines an analytic function of two variables inside the sectors given by

$$(3.5)\qquad |\arg(x)|<\frac{1}{2}\Delta\pi,\quad |\arg(y)|<\frac{1}{2}\nabla\pi.$$

The integral equation (3.1) is, in fact, a particular case of a multidimensional integral transform considered in the paper of Srivastava and Panda [12, p. 119, Eqn. (1.1)], wherein, a fairly detailed description is given regarding the various *sufficient conditions* under which (3.1) exists. Without recording these here, we assume that these conditions hold true in this paper.

We now state our proposed extension of Theorem 1 which is contained in the following:

THEOREM 2. *Let* $\bar{g}(p, q)$ *given by* (3.1) *be such that* $\bar{g}\in\Omega$. *Then, for all real values of* λ *and* μ,

$$(3.6)\quad (-1)^{\lambda+\mu}W_p^{\lambda}W_q^{\mu}\bar{g}(p, q)=\int_0^{\infty}\int_0^{\infty}x^{\lambda}y^{\mu}H_*[(px)^h, (qy)^k]dxdy,$$

where

$$(3.7)\quad H_*[x, y]=H\begin{matrix}0, 0:m_2+1, n_2\ ;m_3+1, n_3\\ p_1, q_1:p_2+1, q_2+1;p_3+1, q_3+1\end{matrix}\left[\begin{matrix}x\\ y\end{matrix}\middle|\begin{matrix}(a_i-\frac{\lambda\alpha_i}{h}-\frac{\mu A_i}{k};\alpha_i, A_i)_{1,p_1} : (c_i-\frac{\lambda\gamma_i}{h}, \gamma_i)_{1,p_2}, (-\lambda, h);\\ (b_i-\frac{\lambda\beta_i}{h}-\frac{\mu B_i}{k};\beta_i, B_i)_{1,q_1} : (0, h), (d_i-\frac{\lambda\delta_i}{h}, \delta_i)_{1,q_2};\end{matrix}\right.$$

$$\left.\begin{matrix}(e_i-\frac{\mu F_i}{h}, F_i)_{1,p_3}, (-\mu, k)\\ (0, k), (f_i-\frac{\mu F_i}{k}, F_i)_{1,q_3}\end{matrix}\right],$$

provided that the integrals involved are absolutely convergent.

4. Proof of Theorem 2

We establish our theorem under the following cases:

Case I. When $\lambda<0$, $\mu<0$.

Under this case, we notice from (1.1) that

$$(4.1)\quad (-1)^{\lambda+\mu}W_p^{\lambda}W_q^{\mu}\bar{g}(p, q)=\frac{1}{\Gamma(-\lambda)\Gamma(-\mu)}\int_p^{\infty}\int_q^{\infty}(u-p)^{-\lambda-1}(v-q)^{-\mu-1}\cdot\left\{\int_0^{\infty}\int_0^{\infty}H^*[(ux)^h, (vy)^k]g(x, y)dxdy\right\}dudv.$$

On changing the order of integrations, (4.1) may be written as

$$(4.2)\quad (-1)^{\lambda+\mu}W_p^{\lambda}W_q^{\mu}\bar{g}(p, q)=\frac{1}{\Gamma(-\lambda)\Gamma(-\mu)}\int_0^{\infty}\int_0^{\infty}g(x, y)\cdot\left\{\int_p^{\infty}\int_q^{\infty}(u-p)^{-\lambda-1}(v-q)^{-\mu-1}H^*[(ux)^h, (vy)^k]dudv\right\}dxdy.$$

The inversion in the order of integrations being justified in view of the assumptions made in the hypothesis of the theorem. Now If I denotes the *inner* double integral in (4.2), then I can easily be evaluated by using the definition of the function H^* from [7, p. 117] and employing simple substitutions (see also the result [11, p. 40, Eqn. (3.3)]). We are led to

$$(4.3)\quad I=p^{-\lambda}q^{-\mu}\Gamma(-\lambda)\Gamma(-\mu)\, H^{0,0:m_2+1,n_2;m_3+1,n_3}_{p_1,q_1:p_2+1,q_2+1;p_3+1,q_3+1}\left[\begin{matrix}(px)^h\\(qy)^k\end{matrix}\middle|\begin{matrix}(a_i;\alpha_i,A_i)_{1,p_1}:(c_i,\gamma_i)_{1,p_2},(0,h);(e_i,F_i)_{1,p_3},(0,k)\\(b_i;\beta_i,B_i)_{1,q_1}:(\lambda,h),(d_i,\delta_i)_{1,q_2};(\mu,k),(f_i,F_i)_{1,q_3}\end{matrix}\right]$$

where $h>0, k>0$, $h\ \mathrm{Re}[(c_i-1)/\gamma_i]<\lambda<0$, $k\ \mathrm{Re}\ [(e_j-1)/E_j]<\mu<0$, for $i=1,\dots,n_2$; $j=1,\dots,n_3$ and $|\arg(p)|<\frac{\pi\Delta}{2h}$, $|\arg(q)|<\frac{\pi\nabla}{2k}$, $\Delta>0, \nabla>0$ (Δ and ∇ being given by (3.3) and (3.4) respectively). With this evaluation of I, (4.2) is seen to reduce to the desired relation (3.6) on modifying the integrand by appealing to certain known identities *Eqn.* (2.1) and the *corrected version* of *Eqn.* (2.2) on p. 120 of [5].

Case II. When $\lambda>0$, $\mu>0$.

By using (1.2), we may express by virtue of Case I,

$$(4.4)\quad (-1)^{\lambda+\mu}W_p{}^{\lambda}W_q{}^{\mu}\bar{g}(p,q)=(-1)^{m+n}\ D^{m+n}_{p,q}\int_0^\infty\int_0^\infty x^{\lambda'}y^{\mu'}g(x,y)$$
$$\cdot H^{0,0:m_2+1,n_2;m_3+1,n_3}_{p_1,q_1:p_2+1,q_2+1;p_3+1,q_3}\left[\begin{matrix}(px)^h\\(qy)^k\end{matrix}\middle|\begin{matrix}(a_i;\alpha_i,A_i)_{1,p_1}\\(b_i;\beta_i,B_i)_{1,q_1}\end{matrix}\right.$$
$$\left.\begin{matrix}(c_i-\frac{\lambda'\gamma_i}{h},\gamma_i)_{1,p_2},(-\lambda',h);(e_i-\frac{\mu'E_i}{k},E_i)_{1,p_3},(-\mu',k)\\(0,h),(d_i-\frac{\lambda'\delta_i}{h},\delta_i)_{1,q_2};(0,k),(f_i-\frac{\mu F_i}{k},F_i)_{1,q_3}\end{matrix}\right]dxdy\Bigg),$$

where, for brevity's sake,

$$(4.5)\qquad \lambda'=\lambda-m,\quad \mu'=\mu-n.$$

Differentiating under the sign of integral and appealing to the formula given by Raina and Koul [9, Eqn. (12) *et seq.*], and using, as before, the identities given in [5], we arrive at the desired result (3.6).

Case III. When $\lambda>0$, $\mu<0$.

In this case, from the defining equations (1.1) and (1.2) we have

$$(4.6)\quad (-1)^{\lambda+\mu}\ W_p^{\lambda} W_q^{\mu} \bar{g}(p,q) = (-1)^{\lambda+\mu} D_p^{m}\Big\{\frac{(-1)^{\lambda+\mu-m}}{\Gamma(m-\lambda)\Gamma(-\mu)} \cdot \int_p^\infty\int_p^\infty (u-p)^{m-\lambda-1}\ (v-q)^{-\mu-1}\Big(\int_0^\infty\int_0^\infty H^*[(ux)^h,(vy)^k]\, f(x,y)\, dxdy\Big) dudv\Big\}.$$

Applying the same analysis, as detailed, in Case I above (for evaluating the expression on the right side), we again arrive at the relation (3.6) for $\lambda>0$, $\mu<0$.

Case IV. When $\lambda<0$, $\mu>0$.

By interchanging the roles of λ and μ and proceeding as in Case III above, the relation (3.9) is readily seen to hold true for this case also.

Case V. When $\lambda=\mu=0$.

By virtue of the identity operator (1.3), the relation (3.6) is evidently satisfied trivially when $\lambda=\mu=0$.

Thus, in view of the different Cases (I–V) discussed above, the desired relation (3.6) holds true for all real values of λ and μ. This completes the proof.

5. Special cases of Theorem 2

At the outset we observe that if we take $p_1=q_1=0$, then the function H_* in (3.7) degenerates into the product of two H-functions of Fox [3]. Consequently, then our Theorem 2 would reduce at once to Theorem 1 when the parameters are chosen suitably and p (or q) tends to zero.

Next if we choose each of the parameters $\alpha_i, \beta_i, A_i, B_i, \gamma_i, \delta_i, E_i, F_i$ equal to 1, and put $h=k=1$, then from Theorem 2, an equivalent result involving the Meijers's G-function of two variables [8, p. 25] can easily be deduced. This result can further be specialized to yield analogous results other known double transforms because of the many particular cases that this specialized integral transform (3.1) with the G-function of two variables in its kernel possesses.

To illustrate, we deduce the following result. Setting $p_1=q_1=0$, $h=k=1$, $m_2=q_2=m_3=q_3=1$, $n_2=p_2=n_3=p_3=0$, $d_1=f_1=0$, $0_1=F_1=1$, the double integral transform (3.1) is seen to reduce to the double

Laplace transform

$$(5.1)\qquad \bar{g}(p,q)=L[g(x,y);p,q]=\int_0^\infty\int_0^\infty e^{-px-qy}\,g(x,y)\,dxdy,$$

where $g(x,y)$ is of that class such that the double integral is absolutely convergent.

By making the above substitutions in (3.6), we get

$$(5.2)\qquad (-1)^{\lambda+\mu}W_p^{\lambda}W_q^{\mu}g(p,q)=L[x^{\lambda}y^{\mu}g(x,y);p,q]^*,$$

valid for all real values of λ and μ.

It is interesting to observe that (5.2) envelops some of the results given in [2] which arise from (5.2) by appropriately using the properties (1.2)−(1.4), when $\lambda=\pm m$, $\mu=\pm n$ (m,n integers).

References

1. M.A. Al-Bassam, *H-R transforms in two dimensions and some of its applications to partial differential equations*, Fractional Calculus and its applications, Lecture Notes in Math. **457**, Springer-Verlag, New York, 1975.
2. V.A. Ditkin, and A.P. Prudnikov, *Operational calculus in two variables and its applications*, Pergamon Press, New York, 1962.
3. C. Fox, *The G and H functions as symmetrical Fourier kernels*, Trans. Amer. Math. Soc. **98**(1961), 395-429.
4. K.C. Gupta, and P.K. Mittal, *The H-function transform*, J. Austral. Math. Soc. **11**(1970), 142-148.
5. S.P. Goyal, *The H-function of two variables*, Kyungpook Math. J. **15** (1975),
6. K.S. Miller, *The Weyl fractional calculus*, Fractional Calculus and its applications, Lecture Notes in Math. **457**, Springer-Verlag, New York, 1975.
7. P.K. Mittal, and K.C. Gupta, *An integral involving a generalized function of two variables*, Proc. Indian Acad. Sci. **A75** (1975), 117-123.
8. A.M. Mathai, and R.K. Saxena, *The H-function with applications in statistics and other disciplines*, Halsted Press, John Wiley and Sons, New York, London, Sydeny and Toronto, 1978.
9. R.K. Raina, and C.L. Koul, *Fractional derivatives of the H-function*,

* The result (5.2) which is a special case of our main result (3.6) is a known result due to R.K. Raina and V.S. Kiryakova [Bulg. Acad. Sci. Comptes Rendus, **36** (1983), to appear].

Jñānabha **A7** (1977), 97-106.
10. R.K. Raina, and C.L. Koul, *On Weyl fractional calculus and H-function transform*, Kyungpook Math. J. **21** (1981), 275-279.
11. R.K. Raina, *On Weyl fractional differentiation*, Pure Appl. Math. Sci. **10** (1979), 37-41.
12. H.M. Srivastava, and R. Panda, *Certain multidimensional integral transformations*, Proc. Nederl. Akad. Wetensch. **A81** (1978), 118-144.
13. H.M. Srivastava, K.C. Gupta, and S.P. Goyal, *The H-function of one & two variables and its applications*, South Asian Publishers, New Delhi, Madras, 1982.

Department of Mathematics
University of Udaipur
(S. K. N. Agriculture College)
Jobner, Rajasthan
INDIA-303329