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REGULAR RELATIONS IN TRANSFORMATION GROUPS

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0. Introduction

The proximal relations in transformation groups have been introduced by R. Ellis and W. H. Gottschalk [8] and studied in [4], [5], [6], [9]. The proximal relation is a reflexive, symmetric and invariant, but is not in general transitive or closed. The proximal relation plays an important role to characterize the transformation groups. In [2], J. Auslander generalized minimal proximal transformation group by defining the regular minimal set.

In this paper we introduce the regular relation as the generalization of the proximal relation. This paper investigates the properties of the regular relations and show some relations between proximality and regularity. We also give some necessary and sufficient conditions for the regular relation and proximal relation to be transitive.

1. Preliminaries

Throughout this paper (X, T) will denote a transformation group with compact Hausdorff phase space X. A closed nonempty subset A of X is called a ***minimal set*** if for every $x\in A$ the orbit xT is a dense subset of A. If X is itself minimal, we say it is a ***minimal transformation group***. The compact Hausdorff space X carries a natural uniformity whose indices are the neighborhoods of the diagonal in $X\times X$. The points x and y of X are called ***proximal*** provided that for each index α of X, there exists a $t\in T$ such that $(xt, yt)\in\alpha$. The ***proximal relation*** of (X, T), denoted by $P(X, T)$ or P, is defined to be the set of all couples $(x, y)\in X\times X$ such that x is proximal to y. The transformation group (X, T) is said to be ***proximal*** if the proximal relation equals $X\times X$.

Let (X, T) and (Y, T) be transformation groups. If π is a continuous map from X to Y with $\pi(xt)=\pi(x)t$ $(x\in X,\ t\in T)$, then π is called

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a *homomorphism. Endomorphism* and *automorphism* are defined similarly. The set of automorphisms is denoted by $A(X)$. A subset A of the phase group T is said to be *syndetic* provided that $T=AK$ for some compact subset K of T. A minimal transformation group is said to be *regular minimal* if for any two points x, x' in X, there is an endomorphism ϕ of X such that $\phi(x), x'$ are proximal.

Given a transformation group (X, T), we may regard T as a set of self-homeomorphisms of X. We define $E(X)$, the *enveloping semigroup* of X, to be the closure of T in X^X, taken with the topology of pointwise convergence. $(E(X), T)$ is at once a transformation group and $E(X)$ admits a semigroup structure. Viewing $E(X)$ as a collection of functions, it has a natural action on X. We let xp denote the point in X thus obtained from $x\in X$ and $p\in E(X)$. The *minimal right ideal* I is the nonempty subset of $E(X)$ with $IE(X)\subset I$, which contains no proper nonempty subsets with the same property.

Let u and v be two idempotents in $E(X)$. Then u and v are said to be *equivalent* if $uv=u$ and $vu=v$.

Ellis [6] showed that the maps $\Theta_x : E(X)\to X$ defined by $\Theta_x(p)=xp$ are homomorphism with range xT. He also showed that given an epimorphism $\pi : X\to Y$ there exists a unique epimorphism $\psi : E(X)\to E(Y)$ such that for each $x\in X$, $\pi\Theta_x=\Theta_{\pi(x)}\psi$.

2. Regular Relations

In this section, we define regular relation $R(X, T)$, which is motivated by the regular minimal transformation group and proximal relation, and we investigate some properties of $R(X, T)$.

DEFINITION 2.2.1. Let (X, T) be a transformation group and $x, y \in X$. Then x and y are *regular* if there exists a ϕ in $A(X)$ such that $(\phi(x), y)\in P(X, T)$.

The set of all regular pairs in X is called the *regular relation* and is denoted by $R(X, T)$, or simply R.

REMARK 2.2.2. (1) If $A(X)$ is the trivial group, then $R(X, T)$ coincides with $P(X, T)$.

(2) Let (X, T) be a minimal transformation group. Then (X, T) is regular minimal if and only if $R(X, T)=X\times X$.

THEOREM 2.2.3. *Let (X, T) be a transformation group. Then the* *following statements hold.*

(1) $R(X, T)$ *is a reflexive, symmetric and invariant relation.*

(2) *If* $E(X)$ *contains just one minimal right ideal, then* $R(X, T)$ *is an equivalence relation.*

Proof. (1) We only show that $R(X, T)$ is symmetric. Let $(x, y) \in R(X, T)$. There exist some minimal right ideal I of $E(X)$ and ϕ in $A(X)$ such that $\phi(x)\, p = yp$ for all $p \in I$. It also follows that $xp = \phi^{-1}(y)p$, which implies that $(y, x) \in R(X, T)$.

(2) It suffices to show that $R(X, T)$ is transitive. Let I be the only minimal right ideal in $E(X)$, and let $(x, y) \in R(X, T)$ and $(y, z) \in R(X, T)$. There exist ϕ, Ψ in $A(X)$ such that $\phi(x)p = yp$ and $\Psi(y)p = zp$ for all $p \in I$. It follows that

$$\Psi\phi(x)p = \Psi(y)p = zp$$

for all $p \in I$. Therefore $(x, z) \in R(X, T)$.

It is well-known that $P(X, T)$ is transitive if and only if there is only one minimal right ideal in $E(X)$. Therefore we immediately obtain the following corollary.

COROLLARY 2.2.4. *If* $P(X, T)$ *is transitive, then so is* $R(X, T)$.

The converse of Corollary 2.2.4 is not always true as the following example shown.

EXAMPLE 2.2.5. Let X be the unit circle in R^2 centered at the origin. A point Q on X will be given the polar coordinates $(1, \alpha)$, where α is the angle between 0 and 2π. We define a homeomorphism f of X onto X by

$$(1, \alpha)f = \begin{cases} (1, \alpha + \sin\alpha), & \text{if } 0 \leq \alpha \leq \pi \\ (1, \alpha + \sin(2\pi - \alpha)), & \text{if } \pi < \alpha \leq 2\pi \end{cases}$$

The group T consists of the positive and negative powers of f. Then (X, T) is a transformation group.

To find all proximal pairs, firstly we show that

$$\lim (1, \alpha)f^n = (1, \pi)$$

for all $0 < \alpha \leq \pi$, where f^n means the composition of f by n times. Put

$$\theta_1 = \alpha + \sin\alpha$$

$$\theta_2=\theta_1+\sin\theta_1$$
$$\cdots$$
$$\cdots$$
$$\cdots$$
$$\theta_n=\theta_{n-1}+\sin\theta_{n-1}$$
$$\cdots$$
$$\cdots$$

Then $(1,\alpha)f^n=(1,\theta_{n-1}+\sin\theta_{n-1})$.
To prove $\lim(1,\alpha)f^n=(1,\pi)$, it suffices to show that $\lim\theta_n=\pi$. Assume that $0<\alpha\leq\pi$. For $n\in N$,

$$\theta_{n+1}=\theta_n+\sin\theta_n$$

Hence we obtain a sequence (θ_n) which is monotone increasing and bounded in $[0,\pi]$. Therefore (θ_n) converges, say $\lim\theta_n=t$, and

$$t=\lim\theta_{n+1}=\lim(\theta_n+\sin\theta_n)=t+\sin(\lim\theta_n)$$

Thus we obtain $\sin(\lim\theta_n)=0$. Since $\alpha>0$, $\lim\theta_n=\pi$. It follows that $\alpha+\sin\alpha+\sin(\alpha+\sin\alpha)+\sin(\alpha+\sin(\alpha+\sin\alpha))+\cdots\cdots\cdots$ converges to π for all $0<\alpha\leq\pi$.
Through the similar verification, we obtain,

$$\begin{aligned}P(X,T)=\Delta&\cup\{((1,\alpha),(1,\beta))\,|\,\alpha\in(0,\pi],\beta\in(0,\pi]\}\\&\cup\{((1,\alpha),(1,\beta))\,|\,\alpha\in[0,\pi),\beta\in[0,\pi)\}\\&\cup\{((1,\alpha),(1,\beta))\,|\,\alpha\in[\pi,2\pi),\beta\in[\pi,2\pi)\}\\&\cup\{((1,\alpha),(1,\beta))\,|\,\alpha\in(\pi,2\pi],\beta\in(\pi,2\pi]\}\end{aligned}$$

Clearly, $P(X,T)$ is not transitive.
Now we show that $R(X,T)$ is transitive. Define $\phi:X\to X$ by

$$\phi(1,\alpha)=(1,\alpha+\pi).$$

Then ϕ is a bijective continuous function. To show that ϕ is an automorphism of X, it suffices to prove that

$$\phi((1,\alpha)f^n)=(\phi(1,\alpha))f^n$$

for $0\leq\alpha\leq\pi$ and $n\in N$, because the other cases can be verified similarly. Since $(1,\alpha)f^n=(1,\theta_{n-1}+\sin\theta_{n-1})$, it follows that

$$\phi((1,\alpha)f^n)=(1,\pi+\theta_{n-1}+\sin\theta_{n-1})$$

and

$$(\phi(1,\alpha))f^n=(1,\alpha+\pi)f^n$$

$$= (1, \pi+\theta_{n-1}+\sin(2\pi-\pi-\theta_{n-1}))$$
$$= (1, \pi+\theta_{n-1}+\sin\theta_{n-1})$$

Thus ϕ is an automorphism of X. We can easily show that $R(X, T)$ is transitive.

In the following theorem, we find the necessary and sufficient conditions for the regular relation to be an equivalence relation.

THEOREM 2.2.6. *Let (X, T) be a transformation group. Then the following statements are equivalent*:

(1) *$R(X, T)$ is an equivalence relation.*

(2) *Let u and v be the equivalent idempotents in any two minimal right ideals of $E(X)$. Then $(xu, xv)\in R(X, T)$ for all $x\in X$.*

Proof. (1) implies (2). Let u and v be the equivalent idempotents in I and K of $E(X)$ respectively. Then $(xu, x)\in R(X, T)$ and $(x, xv)\in R(X, T)$ for every $x\in X$. Since $R(X, T)$ is an equivalence relation, it follows that $(xu, xv)\in R(X, T)$.

(2) implies (1). Let $(x, y)\in R(X, T)$ and $(y, z)\in R(X, T)$. There exist ϕ and Ψ in $A(X)$ such that $(\phi(x), y)\in P(X, T)$ and $(\Psi(y), z)\in P(X, T)$. Therefore, there exist minimal right ideals, I and K in $E(X)$ such that

$$\phi(x)p=yp,\quad \Psi(y)q=zq \tag{1}$$

for all $p\in I$ and $q\in K$.
Since $L_v: I\to K$ defined by $L_v(p)=vp$ is an isomorphism, for $q\in K$, there exists $p_0\in I$ such that $L_v(p_0)=q$. Therefore

$$\Psi(y)vp_0=\Psi(y)q=zq=zvp_0 \tag{2}$$

By hypothesis, $(\Psi(y)u, \Psi(y)v)\in R(X, T)$ and $(zu, zv)\in R(X, T)$. There exist h_1 and h_2 in $A(X)$ such that

$$(h_1\Psi(y)u, \Psi(y)v)\in P(X, T)$$

and

$$(h_2zu, zv)\in P(X, T) \tag{3}$$

Since $(h_1\Psi(y)u, \Psi(y)v)v=(h_1\Psi(y)u, \Psi(y)v)$ and $(h_2zu, zv)v=(h_2zu, zv)$, we obtain

$$(h_1\Psi(y))u=\Psi(y)v \text{ and } h_2zu=zv. \tag{4}$$

Therefore, we have

$$h_1\Psi(y)p_0=h_1\Psi(y)up_0=\Psi(y)vp_0$$
$$=zvp_0=h_2zup_0=h_2zp_0 \tag{5}$$

From (1) and (5),

$$(h_2^{-1}h_1\Psi\phi(x))p_0=(h_2^{-1}h_1\Psi(y))p_0=zp_0 \tag{6}$$

which implies $(h_2^{-1}h_1\Psi\phi(x), z)\in P(X, T)$. This shows that $(x, z)\in R(X, T)$, and hence $R(X, T)$ is transitive. The proof is completed.

COROLLARY 2.2.7. *Let (X, T) be a transformation group. If $R(X, T)$ is a closed subset of $X\times X$, then $R(X, T)$ is an equivalence relation.*

Proof. Let u and v be the equivalent idempotents in any two minimal right ideals of $E(X)$. For every $x\in X$, it follows that $(xu, x)\in R(X, T)$. Since $R(X, T)$ is a closed subset of $X\times X$, we obtain $(xu, xv)=(xu, x)v\in R(X, T)$. This completes the proof.

For any minimal right ideal I of $E(X)$ and $x\in X$, let $\theta_x{}^I : I\to X$ be the map defined by $\theta_x{}^I(p)=xp$ for all $p\in I$. Let u and v be the equivalent idempotents in I and K of $E(X)$, respectively. For every $x\in X$, $(xu, xv)\in R(X, T)$ implies there exists ϕ in $A(X)$ such that $(\phi(xu), xv)\in P(X, T)$. Since $(\phi(xu), xv)$ is an almost periodic point of $(X\times X, T)$, it follows that $\phi(xu)=xv$. That is,

$$\phi\circ\theta_x{}^I(u)=\theta_x{}^K\circ L_v(u).$$

Since I is minimal, we obtain $\phi\circ\theta_x{}^I=\theta_x{}^K\circ L_v$.
On the other hand, let $\phi\circ\theta_x{}^I=\theta_x{}^K\circ L_v$.
Then $\phi(xu)=xvu=xv$, which shows that $(xu, xv)\in R(X, T)$. Therefore we obtain the following corollary.

COROLLARY 2.2.8. *Let $x\in X$, and let u and v be the equivalent idempotents in minimal right ideals I and K of $E(X)$, respectively. The following statements are equivalent:*

(1) $(xu, xv)\in R(X, T)$.

(2) *There exists ϕ in $A(X)$ such that $\phi\circ\theta_x{}^I=\theta_x{}^K\circ L_v$.*

REMARK 2.2.9. It is easily shown that $R(X, T)$ is an equivalence relation if and only if for any two pairs $(x, y)\in P(X, T)$ and $(y, z)\in P(X, T)$, there exists ϕ in $A(X)$ such that $(\phi(x), z)\in P(X, T)$.

For a fixed ϕ in $A(X)$, we denote $R_\phi(X, T)$ or simply R_ϕ to be the set of all $(x, y)\in X\times X$ satisfying the condition $(\phi(x), y)\in P(X,$

T). The set of all $R_\phi(X, T)$ ($\phi \in A(X)$) is denoted by $G(X, T)$.

The following theorem gives us another necessary and sufficient conditions for $P(X, T)$ and $R(X, T)$ to be transitive.

THEOREM 2.2.10. *Let (X, T) be a transformation group. Then the following statements hold*:

(1) *$P(X, T)$ is an equivalence relation if and only if $R_\Psi \circ R_\phi = R_{\Psi\phi}$ for all Ψ, ϕ in $A(X)$.*

(2) *$R(X, T)$ is an equivalence relation if and only if for Ψ, ϕ in $A(X)$, there exists θ in $A(X)$ such that $R_\Psi \circ R_\phi \subset R_\theta$.*

Proof. (1) Necessity. Suppose $(x, z) \in R_\Psi \circ R_\phi$. There exists $y \in X$ such that $(x, y) \in R_\phi$ and $(y, z) \in R_\Psi$. That is, $(\phi(x), y) \in P(X, T)$ and $(\Psi(y), z) \in P(X, T)$. We also have $(\Psi\phi(x), \Psi(y)) \in P(X, T)$. Since $P(X, T)$ is an equivalence relation, it follows that $(\Psi\phi(x), z) \in P(X, T)$, which implies that $(x, z) \in R_{\Psi\phi}$. On the other hand, let $(x, z) \in R_{\Psi\phi}$. Then $(\Psi\phi(x), z) \in P(X, T)$. Put $\phi(x) = y$. We have $(\Psi(y), z) \in P(X, T)$ and $(\phi(x), y) = (y, y) \in P(X, T)$, that is, $(y, z) \in R_\Psi$ and $(x, y) \in R_\phi$. Therefore $(x, z) \in R_\Psi \circ R_\phi$.

Sufficiency. It remains only to show that $P(X, T)$ is transitive. Let $(x, z) \in P(X, T) = R_1(X, T)$ and $(y, z) \in P(X, T) = R_1(X, T)$. By hypothesis, $(x, z) \in R_1 \circ R_1 = R_1 = P(X, T)$, which shows that $P(X, T)$ is transitive.

(2) Necessity. Let $(x, z) \in R_\Psi \circ R_\phi$. There exists $y \in X$ such that $(x, y) \in R_\phi \subset R(X, T)$ and $(y, z) \in R_\Psi \subset R(X, T)$. Since $R(X, T)$ is an equivalence relation, we obtain $(x, z) \in R(X, T)$. Therefore, there exists θ in $A(X)$ such that $(x, z) \in R_\theta$.

Sufficiency. Let $(x, y) \in R(X, T)$ and $(y, z) \in R(X, T)$. There exist ϕ and Ψ in $A(X)$ such that $(x, y) \in R_\phi$ and $(y, z) \in R_\Psi$. Therefore, $(x, z) \in R_\Psi \circ R_\phi \subset R_\theta$ for some θ in $A(X)$. This shows that $(x, z) \in R(X, T)$. The proof is completed.

REMARK 2.2.11. (1) $(R_\phi)^{-1} = R_{\phi^{-1}}$ for all ϕ in $A(X)$.

(2) If $P(X, T)$ is transitive, then $R_\phi \circ P = P \circ R_\phi = R_\phi$ for all ϕ in $A(X)$.

(3) Let (X, T) be a minimal transformation group and let $P(X, T)$ be an equivalence relation. If $\phi \neq \Psi$ in $A(X)$, then R_ϕ and R_Ψ are disjoint.

From Theorem 2.2.10. (1) and Remark 2.2.11. (1) and (2), we have the following corollary.

COROLLARY 2.2.12. *Let (X, T) be a transformation group and let $P(X, T)$ be an equivalence relation. Then $G(X, T)$ is a group under the composition.*

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