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JORDAN AUTOMORPHISMS ON DIRECT SUMS OF SIMPLE RINGS

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1. Introduction

Suppose R is a direct sum of K simple rings and G is a group of automorphisms of R of finite order, $|G|$. If R has no $|G|$-torsion (i. e. $|G|r=0$ implies $r=0$ for all $r\in R$) then Osterburg has shown in [6] that the fixed ring of R under G is a direct sum of at most $|G|K$ simple rings. We shall prove the analogous result when G consists of Jordan automorphisms of R.

2. Preliminaries

Let R and S be rings and T an additive map of R into S. Then T is called a Jordan homomorphism if (i) $(x^2)^T=(x^T)^2$ and (ii) $(xyx)^T=x^Ty^Tx^T$ for all x, y in R. Any additive map satisfying (i) necessarily satisfies (i′) $(xy+yx)^T=x^Ty^T+y^Tx^T$ and if S has no 2-torsion (i. e. $2s=0$ implies $s=0$ for every $s\in S$) then additivity and (i′) imply both (i) and (ii) [c. f. Herstein: Topics in Ring Theory]. As can be readily verified, any Jordan homomorphism T also satisfies $(xyz+zyx)^T=x^Ty^Tz^T+z^Ty^Tx^T$ as well as $[x, [y, z]]^T=[x^T, [y^T, z^T]]$ where $[a, b]=ab-ba$.

Clearly every (associative) homomorphism or anti-homomorphism is a Jordan homomorphism and conversely we have

THEOREM 1. (Herstein [1]) *Every Jordan homomorphism onto a prime ring is either a homorphism or anti-homomorphism.*

As a corollary we have

COROLLARY 2. (Martindale-Montgomery [4]) *Let T be a Jordan isomorphism from R onto S and let P be a prime ideal of R. Then the image of P under T (denoted P^T) is a prime ideal of S and R/P,*

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S/P^T are either isomorphic or anti-isomorphic.

Suppose R is a non-commutative ring with involution *. Define $\tau : R \to R \oplus R$ by $r^\tau = (r, r^*)$. Then τ is a Jordan monomorphism (i. e. τ is a one-to-one Jordan homomorphism) whose image is not a subring of $R \oplus R$. Similarly, there are Jordan homomorphisms whose kernels are not (associative) ideals. This prompts the following definitions.

An additive subgroup A of an associative ring R is called a (***special***) ***Jordan ring*** if a^2, aba are in A whenever $a, b \in A$.

An additive subgroup I of a Jordan ring A is called a (***quadratic***) ***Jordan ideal*** if x^2, xax, axa, $xa + ax$ are in I whenever $x \in I$ and $a \in A$. We write $I \trianglelefteq_J A$.

Every associative ring is a Jordan ring and every ideal of an associative ring is a Jordan ideal. The image of a Jordan homomorphism is a Jordan ring and the kernel of a Jordan homomorphism is a Jordan ideal. Also, the image of a Jordan ideal under a Jordan homomorphism is a Jordan ideal of the image of the Jordan homomorphism.

In [3], McCrimmon has shown that every non-zero Jordan ideal of a semiprime ring contains a nonzero (associative) ideal. Using this fact and Corollary 2, we can characterize Jordan automorphisms on direct products of prime rings. But first we give some examples.

EXAMPLE 1. Let R be a commutative ring and $\mathrm{Mat}_n(R)$ denote the $n \times n$ matrices over R. Then the map $M \to M^\tau$ which takes each matrix to its transpose is an involution and hence a Jordan automorphism.

EXAMPLE 2. Let R be a non-commutative simple ring with involution*. Define $\tau : R \oplus R \to R \oplus R$ by $\tau(a, b) = (a^*, b)$. Then τ is a Jordan automorphism (of order 2) on a direct sum of simple rings which is neither an automorphism nor anti-antomorphism.

EXAMPLE 3. Let R, * be as in example 2. Define $\tau : R \oplus R \to R \oplus R$ by $\tau(a, b) = (b, a^*)$. Then τ, is a Jordan automorphism of order 4.

EXAMPLE 4. Let R, * be as in example 2. Let p be a prime greater than 2. Let $S = R_1 \oplus R_2 \oplus \dots \oplus R_p$ each $R_i = R$. Define $\tau_p : S \to S$ by $\tau_p (r_1, r_2, \dots, r_p) = (r_2^*, r_3^*, \dots, r_p^*, r_1)$; then τ_p is a Jordan automorphism of order p.

We note that in examples 2, 3, 4 we can replace R with any ring and * with any Jordan automorphism on R in order to obtain a Jordan automorphism. The following theorem shows that these are the only Jordan automorphisms on direct products of prime rings.

THEOREM 3. *Let* $R=\prod_{\alpha\in A} S_\alpha$ *where each* S_α *is a prime ring and let* g *be a Jordan automorphism on* R. *Then for each* $\beta\in A$, *there is* $\gamma\in A$ *so that* $S_\beta{}^g=S_\gamma$ *and the restriction of* g *to* S_β *is either an automorphism or anti-automorphism.*

Proof. S_β is the intersection of the prime ideals of R which contain S_β. Therefore the image of S_β under g is an intersection of prime ideals of R by Corollary 2. In particular, $S_\beta{}^g$ is an associative ideal of R.

Let $K\subseteq A$ such that $S_\beta{}^g$ contains an element which has a non-zero entry in S_l if and only if $l\in K$.

Suppose $l\in K$ and $s\in S_l$ such that s appears in the l^{th} component of an element of $S_\beta{}^g$ with $s\neq 0$. Since $S_\beta{}^g$ is an ideal of R, $S_\beta{}^g S_l\subseteq S_\beta{}^g$. Consequently $sS_l\subseteq S_\beta{}^g$. Likewise $S_l s\subseteq S_\beta{}^g$ and $xsy\in S_\beta{}^g$ for all $x, y\in S_l$. Therefore $S_\beta{}^g$ contains an ideal of S_l, which is non-zero by the primeness of S_l.

So for each $l\in K$ there is a nonzero ideal I_l of S_l such that $S_\beta{}^g \supseteq \prod_{l\in K} I_l$.

We now show that K contains exactly one element. Suppose $l, l' \in K$, $l\neq l'$. Then $I_l\cap I_{l'}=0$. Thus $0=g^{-1}(I_l\cap I_{l'})=g^{-1}(I_l)\cap g^{-1}(I_{l'})$. But $g^{-1}(I_l)$, $g^{-1}(I_{l'})$ are nonzero Jordan ideals of S_β. By McCrimmon's result there are nonzero ideals A and B of S_β with $A\subseteq g^{-1}(I)$ and $B\subseteq g^{-1}(I_{l'})$. But this forces $A\cap B=0$ which contradicts the primeness of S_β. Thus K contains exactly one element. This implies that there is a $\gamma\in A$ such that $S_\beta{}^g\subseteq S_\gamma$.

Applying the same argument to S_γ and g^{-1}, we get $S_\gamma{}^{g^{-1}}\subseteq S_\delta$ where $\delta\in A$. But this implies that $S_\gamma\subseteq S_\delta{}^g$ and so $S_\beta{}^g\subseteq S_\gamma\subseteq S_\delta{}^g$ which gives $S_\beta\subseteq S_\delta$ forcing $\beta=\delta$. Consequently $S_\beta{}^g=S_\gamma$.

The last statement of the theorem is a consequence of Theorem 1.

If R is a ring and G is a group of Jordan automorphisms of R, then the fixed ring of R under G is $\{r\in R \mid r^g=r$ for every $g\in G\}$ and is denoted R^G. If g is an element of any group, then $\langle g\rangle$ denotes the subgroup generated by g. In particular, if I is a subring of R which is g-invariant (i. e. $I^g=I$) then $I^{\langle g\rangle}$ denotes the set of elements of I which are fixed by g.

Finally, we note another consequence of Theorem 1 which appears in [4]. Namely, if R is a prime ring, G is a group of Jordan automorphisms of R, and H is the subgroup of G consisting of (associative) automorphisms of R, then the index of H in G is either one or two. In either case, H is normal in G. If $H \neq G$ then G/H acts as involution on R^H.

3. Main Theorem

In this section we consider the action of a finite group of Jordan automorphisms on a ring which is a direct sum of simple rings. Our main result (Theorem 11) extends theorems of both Osterburg [6] and Sundstrom [7]. We start with the result in [6].

THEOREM 4 [Osterburg]. *Let R be a ring which is the direct sum of k simple rings and G a finite group of automorphisms of R such that R has no $|G|$-torsion. Then the fixed ring of R is a direct sum of l simple rings where $l \leq k|G|$.*

For involutions, we have the following result proved in [1].

THEOREM 5. *If R is a simple ring of characteristic not 2 and g is an involution on R, then the fixed ring of R is a simple Jordan ring.*

A simple Jordan ring is a Jordan ring which has no nonzero proper Jordan ideals. Since every associative ideal is a Jordan ideal, any associative ring which is a simple Jordan ring is a simple ring. Conversely, if R is a simple ring then R is a simple Jordan ring. For if R has a nontrivial proper Jordan ideal, A, then by McCrimmon's result R contains a nonzero associative ideal contained in A. which contradicts the simplicity of R.

In [7], Sundstrom considers the situation when G is a finite solvable group consisting of automorphisms or anti-automorphisms, acting on a direct sum of simple rings which has no $|G|$-torsion. The subgroup of automorphisms of G is a normal subgroup, H, of index 2 with G/H acting on R^H as an involution. In general, when G is a finite solvable group of Jordan automorphisms on a direct sum of simple rings, the subgroup of automorphisms is not necessarily of index 2 in G. In example 3, τ is a Jordan automorphism of order 4 and the only automorphism in $\langle\tau\rangle$ is the identity. When G is not solvable, the subgroup of automorphisms is not necessarily normal as the next example illustrates.

EXAMPLE 5. Let R be a simple non-commutative ring with involution $*$. Let $S=R\oplus R\oplus R$ and $G=\langle\tau,\rho\rangle$ where $\tau(a,b,c)=(a,c^*,b)$ and $\rho(a,b,c)=(c,a,b)$ then $\tau\rho\tau^{-1}(a,b,c)=(b^*,c^*,a)$ so $\tau\rho\tau^{-1}$ is not an automorphism and hence the subgroup of automorphisms of G is not normal.

Nevertheless, by using Theorems 3, 4, and 5, we can prove analogous results for Jordan automorphisms.

We start with

LEMMA 6. *Suppose* $R=\sum_{i=0}^{n-1}\oplus S_i$ *and* g *is a Jordan automorphism of* R *such that*

(i) g^n *is the identity*

and

(ii) $S_i^g=S_{i+1(\text{mod } n)}$

Then the fixed ring of R *is Jordan isomorphic to* S_0.

Proof. If $s\in S_0$ then $s\oplus s^g\oplus s^{g^2}\oplus\ldots\oplus s^{g^{n-1}}$ is fixed by g. Conversely, if $r\in R$ is fixed by g then r is of the form $s\oplus s^g\oplus s^{g^2}\oplus\ldots\oplus s^{g^{n-1}}$ where $s\in S_0$. So $R^{<g>}=\{s\oplus s^g\oplus s^{g^2}\oplus\ldots\oplus s^{g^{n-1}}\mid s\in S_0\}$. The map from S_0 to $R^{<g>}$ given by $s\to s\oplus s^g\oplus s^{g^2}\oplus\ldots\oplus s^{g^{n-1}}$ is a Jordan isomorphism.

We generalize the result in

LEMMA 7. *Suppose* R *is a ring and* g *is a Jordan automorphism such that* $R=\sum_{i=0}^{n-1}\oplus I^{g^i}$. *Then the fixed ring of* R *is Jordan isomorphic to the fixed ring of* I *under* $\langle g^n\rangle$.

Proof. Clearly, each I^{g^i} is g^n-invariant so

$$R^{<g^n>}=(\sum_{i=0}^{n-1}\oplus I^{g^i})^{<g^n>}=\sum_{i=0}^{n-1}\oplus(I^{g^i})^{<g^n>}$$

By letting $S_i=(I^{g^i})^{<g^n>}$ and g' be a generator of $\langle g\rangle/\langle g^n\rangle$ we can apply Lemma 6 to $\sum_{i=0}^{n-1}\oplus S_i$ and g' to obtain

$$(\sum_{i=0}^{n-1}\oplus(I^{g^i})^{<g^n>})^{<g>/<g^n>}=(\sum_{i=0}^{n-1}\oplus S_i)^{<g'>}\cong S_0=I^{<g^n>}$$

But $$(\sum_{i=0}^{n-1}\oplus(I^{g^i})^{<g^n>})^{<g>/<g^n>}=(R^{<g^n>})^{<g>/<g^n>}=R^{<g>}$$

So $R^{<g>}\cong_J I^{<g^n>}$.

We now prove

THEOREM 8. *Let R be a simple ring and G a finite group of Jordan automorphisms of R. If R has no $|G|$-torsion, then the fixed ring of R is a direct sum of at most $|G|$ simple Jordan rings. If, in addition, G does not consist solely of automorphisms then the fixed ring of R is a direct sum of at most $|G|/2$ simple Jordan rings.*

Proof. Let $H=\{g\in G|g$ is an automorphism of $R\}$. If $H=G$, then by Theorem 4 we are done.

We now assume $H\neq G$. Then the index of H in G is equal to 2 and G/H acts as involution of R^H. By Theorem 4, R^H is a direct sum of at most $|H|$ simple rings; so suppose $R^H=\sum_{i=1}^{n}\oplus S_i$ where each S_i is a simple ring and $n\leq|H|$.

We first consider the case when S_1 is G/H-invariant. Either the action of G/H on S_1 is that of the identity or that of an involution. In either case, $S_1^{G/H}$ is a simple Jordan ring.

Now suppose S_1 is not G/H invariant. Then by Theorem 3, there is an $l\leq n$ that the image of S_1 under the non-identity element of G/H is S_l. In this case G/H acts on $S_1\oplus S_l$ and by Lemma 6, $(S_1\oplus S_l)^{G/H}\cong_J S_1$. Continuing, we see that $R^G=(R^H)^{G/H}$ is a direct sum of at most $|H|=|G|/2$ slmple Jordan rings.

We now investigate the situation when R is a direct sum of simple rings, proving first a result about associative automorphisms.

LEMMA 9. *Let $R=S_1\oplus S_2$ where S_1, S_2 are simple rings and let G be a finite group of automorphisms of R. If R has no $|G|/2$ torsion and S_1 is not G-invariant, then the fixed ring of R is a direct sum of at most $|G|/2$ simple rings.*

Proof. Let $K=\{g\in G|S_1^g=S_1\}$. Then K is normal in G and has index 2. Consequently,

$$R^G=(R^K)^{G/K}=((S_1\oplus S_2)^K)^{G/K}=(S_1^K\oplus S_2^K)^{G/K}$$

with is isomorphic to S_1^K by Lemma 6. And by Theorem 4, S_1^K is a direct sum of at most $|K|$ simple rings.

Therefore, R^G is a direct sum of at most $|G|/2$ simple rings.

We now extend this Lemma by allowing Jordan automorphisms.

THEOREM 10. *Let* $R=S_1\oplus S_2$ *where* S_1, S_2 *are simple rings and let* G *be a finite group of Jordan automorphisms of* R *such that* R *has no* $|G|$*-torsion.*

(i) *If* G *does not consist solely of automorphisms, then the fixed ring is a direct sum of at most* $3|G|/2$ *simple Jordan rings.*

(ii) *If* S_1 *is not* G*-invariant, then the fixed ring is a direct sum of at most* $|G|/2$ *simple Jordan rings.*

(iii) *If the hypotheses of (i) and (ii) are both satisfied and* $\{g\in G \mid S_1{}^g=S_1\} \neq \{g\in G \mid g$ *is an automorphism*$\}$, *then the fixed ring is a direct sum of at most* $|G|/4$ *simple Jordan rings.*

Proof. Let $K=\{g\in G \mid S_1{}^g=S_1\}$ and Let $H=\{g\in G \mid g$ is an automorphism of $R.\}$ We first prove:

(ii) Suppose S_1 is not G-invariant. Then the index of K in G is equal to 2. As in the proof of Lemma 9, $R^G\cong_J S_1{}^K$ and by Theorem 8, $S_1{}^K$ is a direct sum of at most $|K|=|G|/2$ simple Jordan rings.

(i) We may assume that S_1 is G-invariant. Otherwise, we can apply part (ii). Since G does not consist solely of automorphisms, its action on either S_1 or S_2 is not that of associative automorphisms. Therefore the fixed ring of either S_1 or S_2 is a direct sum of at most $|G|/2$ simple Jordan rings by Theorem 8. The fixed ring of the other summand is a direct sum of at most $|G|$ simple Jordan rings also by Theorem 8. Therefore R^G is a direct sum of at most $|G|/2+|G|=3|G|/2$ simple Jordan rings.

(iii) As in the proof of Lemma 9, $R^G\cong_J S_1{}^K$ (or, equivalently, $R^G\cong_J S_2{}^K$). If the action of K on both S_1 and S_2 is that of automorphisms then $K\subseteq H$. But the index of K in G is equal to 2. So either $K=H$ or $H=G$. But, by hypothesis, neither of these can happen. Consequently, we may assume that K does not act as automorphism on S_1. By applying Theorem 8, $S_1{}^K$ is a direct sum of at most $|K|/2$ simple Jordan rings. That is, R^G is a direct sum of at most $|K|/2=|G|/4$ simple Jordan rings.

We remark that when S_1 is not G-invariant, we need only require that R has no $|G|/2$ torsion.

As a final result we prove:

THEOREM 11. *Let* R *be a direct sum of* K *simple rings and* G *a finite group of Jordan automorphism of* R *such that* R *has no* $|G|$*-torsion. Then the fixed ring is a direct sum of at most* $K|G|$ *simple*

Jordan rings. This bound can be achieved only when each summand of R is G–invariant and G consists solely of automorphisms of R.

Proof. Let $R=\sum_{i=1}^{K}\oplus S_i$. If each S_i is G–invariant then we can apply Theorem 8 to conclude that R^G is a direct sum of at most $K|G|$ simple Jordan rings. If, in addition, G does not consist solely of automorphisms, then the action of G on some S_l does not act as automorphisms. Consequently, S_l^G is a direct sum of at most $|G|/2$ simple rings. Therefore R^G is a direct sum of at most $(K-1)|G|+|G|/2<K|G|$ simple Jordan rings.

Now assume that some S_l is not G–invariant and let $\text{Orbit}(S_l)=\{S_l^g\,|\,g\in G\}$. We will show that if R' is the direct sum of the distinct elements of Orbit (S_l) then the fixed ring R' under G is a direct sum of at most $|G|/n$ simple Jordan rings where $n=|\text{Orbit}(S_l)|$.

Let $H=\{g\in G\,|\,S_l^g=S_l\}$ and let $g_0, g_1, \ldots, g_{n-1}$ be distinct representatives of the right cosets of h in G (where g_0 is the identity) then $R'=S_l^{g_0}\oplus S_l^{g_1}\oplus\ldots\oplus S_l^{g_{n-1}}$ and $n=[G:H]$.

We claim that $(R')^G=\{s+s^{g_1}+\ldots+s^{g_{n-1}}\,|\,s\in S_l^H\}$. Clearly any element of $(R')^G$ is of the form $s+s^{g_1}+\ldots+s^{g_{n-1}}$ where $s\in S^H$. Now let $g\in G$. Then $(s+s^{g_1}+\ldots+s^{g_{n-1}})^g=s^g+s^{g_1g}+\ldots+s^{g_{n-1}g}$. But there is a $h\in H$ and $g_{i_0}(0\leq i_0\leq n-1)$ so that $g=hg_{i_0}$. Consequently, $s^g=s^{hg_{i_0}}=s^{g_{i_0}}$. Similarly, there is $h'\in H$ and g_{i_1} $(0\leq i_1\leq n-1)$ so that $g_1g=h'g_{i_1}$. Therefore $s^{g_1g}=s^{h'g_{i_1}}=s^{g_{i_1}}$ Continuing, we see that the action of $g\in G$ simply permutes the elements of $s+s^{g_1}+\ldots+s^{g_{n-1}}$. We need only show that $\{g_{i_0}, g_{i_1}, \ldots, g_{i_{n-1}}\}$ are distinct representatives of the right cosets of H in G.

Suppose g_{i_α} and g_{i_β} are in the same right coset. Then there is a $g_j(0\leq j\leq n-1)$ and $h_1, h_2\in H$ so that $g_{i_\alpha}=h_1g_j$ and $g_{i_\beta}=h_2g_j$. From before, there exists h', h'' so that $g_\alpha g=h'g_{i_\alpha}$ and $g_\beta g=h''g_i$ where g_α, g_β are distinct in $\{g_0, g_1, \ldots, g_{n-1}\}$. Consequently, $g_\alpha g=h'g_{i_\alpha}=h'h_1g_j$ and $g_\beta g=h''g_ig_j$. That is, $g_\alpha=h'h_1g_jg^{-1}$ and $g_\beta=h''h_2g_jg^{-1}$ which puts g_α and g_β in the same right coset of H in G, a contradiction.

Thus, $\{g_{i_0}, g_{i_1}, \ldots, g_{i_{n-1}}\}$ are distinct representatives of the right cosets of H in G.

We have shown that

$$(R')^G=\{s+s^{g_1}+\ldots+s^{g_{n-1}}\,|\,s\in S^H\}.$$

But the mapping of S_l^H onto $(R')^G$ given by $s\rightarrow s+s^{g_1}+\ldots+s^{g_{n-1}}$ is a

Jordan isomorphism. That is, $(R')^G \cong_J S_l^H$. But by Theorem 8, S_l^H is a direct sum of at most $|H| = |G|/n$ simple Jordan rings. This completes the proof.

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