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NECESSARY AND SUFFICIENT CONDITIONS FOR THE FRESNEL INTEGRABILITY OF CERTAIN CLASSES OF FUNCTIONS

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1. Some preliminaries and further introductory remarks

Let H be a separable Hilbert space over $\mathbf{R}$ (the real numbers). Let $M(H)$ denote the space of $\mathbf{C}$-valued (or complex-valued), countably additive, Borel measures μ on H of finite total variation $\|\mu\|$. With convolution taken as multiplication, $M(H)$ is a commutative Banach algebra with identity referred to as the measure algebra of H. The Fourier transform of an element μ of $M(H)$ is defined for all γ in H by

$$\hat{\mu}(\gamma):=\int_H \exp\{i(\gamma,h)\}\,d\mu(h). \tag{1}$$

The space $\mathcal{F}(H)$ of Fresnel integrable functions on H is just the space of all Fourier transforms $\hat{\mu}$ where μ is in $M(H)$. The map $\mu\to\hat{\mu}$ is one-to-one. Using this fact, $\|\hat{\mu}\|$ is defined as the total variation $\|\mu\|$ of μ. Under this norm and with pointwise multiplication of functions, $\mathcal{F}(H)$ is a commutative Banach algebra with identity.

We are especially interested in the particular Hilbert space appropriate for ordinary quantum mechanics. For simplicity we will restrict ourselves to one space dimension. Fix $t>0$. Let H_t be the space of $\mathbf{R}$-valued functions γ on $[0,t]$ which are absolutely continuous with square integrable derivative $D\gamma$ and which satisfy $\gamma(t)=0$ H_t is a separable Hilbert space with inner product

$$(\gamma_1,\gamma_2):=\int_0^t (D\gamma_1)(s)(D\gamma_2)(s)\,ds. \tag{2}$$

The spaces $\mathcal{F}(H)$ play a key role throughout the fundamental monograph [1] of Albeverio and Høegh-Krohn. The matters above are all discussed in [1]. A good discussion of C-valued measures can be found in [14]. The spaces $\mathcal{F}(H)$ are intimately related to certain

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Banach algebras of functions on Wiener space or abstract Wiener space as has been shown in [8, 13]. One or another of these Banach algebras have played a key role in several papers [1–13, 15, 16]. The study of the appropriate Banach algebra of functions on Wiener space was begun by Cameron and Storvick in [4].

We finish this section with a brief discussion of the concrete problem which motivated this paper.

In dealing with the evolution of a 1-dimensional quantum mechanical system, functions on H_t of the form

$$f(\gamma)=\phi(\gamma(0)) \tag{3}$$

are involved where $\phi:\mathbf{R}\to\mathbf{C}$ is the initial probability amplitude. A simple result of Albeverio and Høegh-Krohn [1; p. 29] shows that if $\phi=\hat{\nu}$ where ν is in the measure algebra $M(\mathbf{R})$ of $\mathbf{R}$, then f is in $\mathcal{F}(H_t)$. We conjectured that no other f's of the form (3) are in $\mathcal{F}(H_t)$; that is, if f is in $\mathcal{F}(H_t)$, then there exists ν in $M(\mathbf{R})$ such that $\phi=\hat{\nu}$ on $\mathbf{R}$. This conjecture turned out to be true and rather easy to prove, and, in the process of looking into it, some more general facts were proved.

REMARK. We actually began by looking at a function analogous to (3) in the setting of a Banach algebra S of functions on Wiener space. We successfully proved that the appropriate sufficient condition for membership in S is also necessary, but our proof is considerably harder than the proof that will be found below. This is of interest because the Banach algebras S and $\mathcal{F}(H_t)$ are known to be equivalent [8], and a number of proofs have been carried over from one setting to the other with only rather minor adjustments. The result in the Wiener space setting will be discussed in a later paper.

We assume throughout that H is a separable, infinite-dimensional Hilbert space over $\mathbf{R}$.

2. The sufficient condition extended

We begin with a lemma which will be useful in this section and the next. We omit the simple proof.

LEMMA 1. *Let $\{\gamma_1, \ldots, \gamma_n\}$ be any finite subset of H. Let $A: H\to \mathbf{R}^n$ be defined by*

(4) $$A\gamma=((\gamma,\gamma_1),\ldots,(\gamma,\gamma_n)).$$

Then A is a bounded linear operator from H to $\mathbf{R}^n$ whose restriction to the span $[\gamma_1,\ldots,\gamma_n]$ of $\{\gamma_1,\ldots\gamma\}$ is one-to-one.

If we let $L(\gamma_1,\ldots,\gamma_n)=L$ be the range of A, then L satisfies

(5) $$L=\{((\gamma,\gamma_1),\ldots,(\gamma,\gamma_n)) : \gamma \text{ is in } [\gamma_1,\ldots,\gamma_n]\}$$

and the dimension of L equals the dimension of $[\gamma_1,\ldots,\gamma_n]$.

The next result is our extended sufficient condition.

THEOREM 2. *Let $\{\gamma_1,\ldots,\gamma_n\}$ be any finite subset of H. Let L be the range of the operator A defined by (4). Let ν be in $M(L)$, the measure algebra of L. Define $f : H\to\mathbf{C}$ by*

(6) $$f(\gamma) := \hat{\nu}((\gamma,\gamma_1),\ldots,(\gamma,\gamma_n)).$$

Then f is in $\mathcal{F}(H)$; in fact, there exists a unique measure μ in $M(H)$ such that

(7) $$\hat{\mu}(\gamma)=\hat{\nu}((\gamma,\gamma_1),\ldots,(\gamma,\gamma_n))$$

for all γ in H. Further μ is supported by $[\gamma_1,\ldots,\gamma_n]$.

Proof. The uniqueness of a measure μ satisfying (7) is a consequence of the fact that the map $\mu\to\hat{\mu}$ is one-to-one. Let $\phi : \mathbf{R}^n\to H$ be defined by $\phi(r_1,\ldots,r_n)=r_1\gamma_1+\ldots+r_n\gamma_n$. ν may be considered as a measure on $\mathbf{R}^n$ supported by L. Let $\mu := \nu\circ\phi^{-1}$. Then μ is in $M(H)$ and, for any γ in H, we can write

$$\begin{aligned}\hat{\mu}(\gamma)&=\int_H \exp\{i(\gamma,h)\}\,d\mu(h)=\int_H \exp\{i(\gamma,h)\}\,d(\nu\circ\phi^{-1})(h)\\ &=\int_{\mathbf{R}^n}\exp\{i(\gamma,\phi(r_1,\ldots,r_n))\}\,d\nu(r_1,\ldots,r_n)\\ &=\int_{\mathbf{R}^n}\exp\{i(\gamma,r_1\gamma_1+\ldots+r_n\gamma_n)\}\,d\nu(r_1,\ldots,r_n)\\ &=\int_{\mathbf{R}^n}\exp\{i(((\gamma,\gamma_1),\ldots,(\gamma,\gamma_n)),(r_1,\ldots,r_n))\}\,d\nu(r_1,\ldots,r_n)\\ &=\int_L \exp\{i(((\gamma,\gamma_1),\ldots,(\gamma,\gamma_n)),(r_1,\ldots,r_n))\,d\nu(r_1,\ldots,r_n)\\ &=\hat{\nu}((\gamma,\gamma_1),\ldots,(\gamma,\gamma_n)).\end{aligned}$$

REMARK 3. If $\gamma_1,\ldots,\gamma_n$ are linearly independent, then, by Lemma 1, dimension (L)=dimension $[\gamma_1,\ldots,\gamma_n]=n$, and so $L=\mathbf{R}^n$.

We need the following well-known result in order to give a corollary of Theorem 2 for the special space H_t defined earlier.

LEMMA 4. *Let* $0 \leq \tau < t$. *Let* $\gamma_\tau(s) := \begin{cases} t-\tau, & 0 \leq s \leq \tau \\ t-\tau-(s-\tau), & \tau \leq s \leq t. \end{cases}$
Then γ_τ *is in* H_t *and* $\gamma(\tau) = (\gamma, \gamma_\tau)$ *for all* γ *in* H_t.

Proof. $(\gamma, \gamma_\tau) = \int_0^t (D\gamma_\tau)(s)(D\gamma)(s)ds = -\int_\tau^t (D\gamma)(s)ds = -\gamma(t) + \gamma(\tau)$
$= \gamma(\tau)$.

COROLLARY 5. *Let* $0 \leq t_1 < \ldots < t_n < t$ *and let* ν *be in* $M(\mathbf{R}^n)$. *Define* $f : H_t \to C$ *by*

$$f(\gamma) := \hat{\nu}(\gamma(t_1), \ldots, \gamma(t_n)). \tag{8}$$

Then f *is in* $\mathcal{F}(H_t)$; *in fact, there exists a unique measure* μ *in* $M(H)$ *such that*

$$\hat{\mu}(\gamma) = \hat{\nu}(\gamma(t_1), \ldots, \gamma(t_n)) \tag{9}$$

for all γ *in* H.

Proof. Let $\gamma_j := \gamma_{t_j}$, $j = 1, \ldots, n$ as in the preceding Lemma. Then $f(\gamma) = \hat{\nu}((\gamma, \gamma_1), \ldots, (\gamma, \gamma_n))$ by Lemma 4. Also it is easy to see that $\gamma_1, \ldots, \gamma_n$ are linearly independent. The result follows from Theorem 2.

Note that the simple result of Albeverio and Høegh-Krohn [1; p. 29] referred to in Section 1 is just the special case of the Corollary with $n=1$ and $t_1=0$.

3. The sufficient condition is also necessary

We begin by showing that the sufficient condition of Theorem 2 is also necessary in the special case where $\{\gamma_1, \ldots, \gamma_n\}$ is an orthonormal set (ONS).

PROPOSITION 6. *Let* $\{\gamma_1, \ldots, \gamma_n\}$ *be any finite* ONS *in* H. *Let* $\phi : \mathbf{R}^n \to \mathbf{C}$. *Suppose that there exist* μ *in* $M(H)$ *such that for every* γ *in* H

$$\hat{\mu}(\gamma) = \phi((\gamma, \gamma_1), \ldots, (\gamma, \gamma_n)). \tag{10}$$

Then there exists a unique measure ν *in* $M(\mathbf{R}^n)$ *such that* $\hat{\nu} = \phi$ *on* $\mathbf{R}^n$.

Proof. Let $A : H \to \mathbf{R}^n$ be given by (4) and let $\nu = \mu \circ A^{-1}$. Using the Change of Variables Theorem and the hypothesis (10) to justify the 3rd and 7th equalities respectively, we can write

$$
\begin{aligned}
\hat{\nu}(r_1, \ldots, r_n) &= \int_{\mathbf{R}^n} \exp\{i((r_1, \ldots, r_n), (s_1, \ldots, s_n))\}\, d\nu(s_1, \ldots, s_n) \\
&= \int_{\mathbf{R}^n} \exp\{i((r_1, \ldots, r_n), (s_1, \ldots, s_n))\}\, d(\mu\circ A^{-1})(s_1, \ldots, s_n) \\
&= \int_H \exp\{i((r_1, \ldots, r_n), A\gamma)\}\, d\mu(\gamma) \\
&= \int_H \exp\{i((r_1, \ldots, r_n), ((\gamma, \gamma_1), \ldots, (\gamma, \gamma_n)))\}\, d\mu(\gamma) \\
&= \int_H \exp\{i(r_1\gamma_1 + \ldots + r_n\gamma_n, \gamma)\}\, d\mu(\gamma) \\
&= \hat{\mu}(r_1\gamma_1 + \ldots + r_n\gamma_n) \\
&= \phi((r_1\gamma_1 + \ldots + r_n\gamma_n, \gamma_1), \ldots, (r_1\gamma_1 + \ldots + r_n\gamma_n, \gamma_n)) \\
&= \phi(r_1, \ldots, r_n).
\end{aligned}
$$

We will use the following lemma to prove our result in the case where no special assumptions are placed on the set $\{\gamma_1, \ldots, \gamma_n\}$. The lemma is surely known and is easy to prove.

LEMMA 7. *Let H_1 and H_2 be two finite-dimensional Hilbert spaces and let T be an invertible linear transformation from H_1 onto H_2. Let η be in $M(H_2)$. Then $\hat{\eta}\circ T : H_1 \to \mathbf{C}$ is the Fourier transform of a measure in $M(H_1)$; in fact, $\hat{\eta}\circ T$ is the Fourier transform of the measure $\eta\circ(T^t)^{-1}$ where T^t is the transpose of T.*

THEOREM 8. *Let $\{\gamma_1, \ldots, \gamma_n\}$ be any finite subset of H. Let A be given by (4) and let L be the range of A. Let ϕ be a $\mathbf{C}$-valued function of n variables defined at least on L. Suppose that there exists μ in $M(H)$ such that for every γ* in *H*

$$
\hat{\mu}(\gamma) = \phi((\gamma, \gamma_1), \ldots, (\gamma, \gamma_n)). \tag{11}
$$

Then there exists a unique measure ν in $M(L)$ such that $\hat{\nu} = \phi$ on L.

Proof. Let $\{\alpha_1, \ldots, \alpha_m\}$ be an ONS which is a basis for $[\gamma_1, \ldots, \gamma_n]$. For each $j = 1, \ldots, n$, $\gamma_j = (\gamma_j, \alpha_1)\alpha_1 + \ldots + (\gamma_j, \alpha_m)\alpha_m$. Hence

$$
\begin{aligned}
\hat{\mu}(\gamma) &= \phi((\gamma, \gamma_1), \ldots, (\gamma, \gamma_n)) \\
&= \phi((\gamma, (\gamma_1, \alpha_1)\alpha_1 + \ldots + (\gamma_1, \alpha_m)\alpha_m), \ldots, (\gamma, (\gamma_n, \alpha_1)\alpha_1 + \ldots + (\gamma_n, \\
&\qquad \alpha_m)\alpha_m)) \\
&= \phi((\gamma_1, \alpha_1)(\gamma, \alpha_1) + \ldots + (\gamma_1, \alpha_m)(\gamma, \alpha_m), \ldots, (\gamma_n, \alpha_1)(\gamma, \alpha_1) + \ldots + \\
&\qquad (\gamma_n, \alpha_m)(\gamma, \alpha_m)) \\
&= \phi\circ B((\gamma, \alpha_1), \ldots, (\gamma, \alpha_m))
\end{aligned}
$$

where B is the linear map from $\mathbf{R}^m$ onto L sending $((\gamma, \alpha_1), \ldots, (\gamma,$

$\alpha_m))$ to $((\gamma_1, \alpha_1)(\gamma, \alpha_1)+\ldots+(\gamma_1, \alpha_m)(\gamma, \alpha_m), \ldots, (\gamma_n, \alpha_1)(\gamma, \alpha_1)+\ldots+(\gamma_n, \alpha_m)(\gamma, \alpha_m))=((\gamma, \gamma_1), \ldots, (\gamma, \gamma_n))$.

By Proposition 6, there exists η in $M(\mathbf{R}^n)$ such that $\hat{\eta}=\phi\circ B$ on $\mathbf{R}^m$. Applying Lemma 7 with $T=B^{-1}$, we see that $\phi=(\phi\circ B)\circ B^{-1}$ is the transform of some measure ν in $M(L)$; that is, $\phi=\hat{\nu}$ for some ν in $M(L)$.

Using Remark 3 we obtain the following corollary.

COROLLARY 9. *Let* $\{\gamma, \ldots, \gamma_n\}$ *be a linearly independent subset of* H. *Let* $\phi : \mathbf{R}^n \to \mathbf{C}$. *Suppose that there exists* μ *in* $M(H)$ *such that* (11) *holds for every* γ *in* H. *Then there exists a unique measure* ν *in* $M(\mathbf{R}^n)$ *such that* $\hat{\nu}=\phi$ *on* $\mathbf{R}^n$.

Next we give a corollary for the space H_t.

COROLLARY 10. *Let* $0\leq t_1<t_2<\ldots<t_n<t$. *Let* $\phi : \mathbf{R}^n\to\mathbf{C}$. *Suppose that there exists* μ *in* $M(H_t)$ *such that for all* γ *in* H_t

$$\hat{\mu}(\gamma)=\phi(\gamma(t_1), \ldots, \gamma(t_n)). \tag{12}$$

Then there exists a unique measure ν *in* $M(\mathbf{R}^n)$ *such that* $\hat{\nu}=\phi$ *on* $\mathbf{R}^n$.

Proof. Letting γ_j, $j=1, \ldots, n$ be defined as in the proof of Corollary 5, the result follows from Corollary 9 and the linear independence of $\gamma_1, \ldots, \gamma_n$ noted earlier.

REMARK 11. The original conjecture referred to in the introduction is settled by the special case of Corollary 10 with $n=1$ and $t_1=0$.

Theorems 2 and 8 as well as Corollaries 5 and 10 can be combined to give necessary and sufficient conditions. The next Corollary does that for Theorems 2 and 8.

COROLLARY 12. *Let* $\{\gamma_1, \ldots, \gamma_n\}$ *be any finite subset of* H. *Let* A *be given by* (4) *and let* L *be the range of* A. *Let* $\phi : L\to\mathbf{C}$. *Define* $f : H\to\mathbf{C}$ *by*

$$f(\gamma) := \phi((\gamma, \gamma_1), \ldots, (\gamma, \gamma_n)). \tag{13}$$

Then f *is in* $\mathcal{F}(H)$ *if and only if* ϕ *is the Fourier transform of a measure* ν *in* $M(L)$.

The results above have all involved relating, under certain conditions, a measure μ on the infinite-dimensional space H to a measure ν on a finite-dimensional space. We finish with a result in the same spirit which, however, involves a measure ν on a subspace K of H which is not necessarily finite-dimensional.

THEOREM 13. *Let K be a closed subspace of H and let P be the orthogonal projection of H onto K. Let $\phi : K \to \mathbf{C}$. Define $f : H \to \mathbf{C}$ by*

$$f(\gamma) := \phi(P\gamma). \tag{14}$$

Then f is in $\mathcal{F}(H)$ if and only if ϕ is in $\mathcal{F}(K)$.

Proof. First suppose that $\phi = \hat{\nu}$ where ν is in $M(K)$. Let $E : K \to H$ be the natural embedding of K into H. Let $\mu := \nu \circ E^{-1}$. Certainly μ is in $M(H)$ and μ is supported by K. To finish the first part of the proof, it suffices to show that $\hat{\mu}(\gamma) = \hat{\nu}(P\gamma)$ for all γ in H. Let γ belong to H.

$$\begin{aligned}\hat{\mu}(\gamma) &= \int_H \exp\{i(\gamma, h)\}\, d\mu(h) = \int_H \exp\{i(\gamma, h)\}\, d(\nu \circ E^{-1})(h)\\ &= \int_K \exp\{i(\gamma, Ek)\}\, d\nu(k) = \int_K \exp\{i(P\gamma + (I-P)\gamma, Ek)\}\, d\nu(k)\\ &= \int_K \exp\{i(P\gamma, Ek)\}\, d\nu(k) = \int_K \exp\{i(P\gamma, k)_K\}\, d\nu(k)\\ &= \hat{\nu}(P\gamma).\end{aligned}$$

Going in the other direction, suppose that there exists μ in $M(H)$ such that $\hat{\mu}(\gamma) = \phi(P\gamma)$ for all γ in H. We must show that there exists ν in $M(K)$ such that $\hat{\nu}(\gamma) = \phi(\gamma)$ for all γ in K. Let $\nu := \mu \circ p^{-1}$ and let γ be in K. Then

$$\begin{aligned}\hat{\nu}(\gamma) &= \int_K \exp\{i(\gamma, k)_K\}\, d\nu(k) = \int_K \exp\{i(\gamma, k)_K\}\, d(\mu \circ p^{-1})(k)\\ &= \int_H \exp\{i(\gamma, ph)_K\}\, d\mu(h) = \int_H \exp\{i(\gamma, ph)\}\, d\mu(h)\\ &= \int_H \exp\{(p\gamma, h)\}\, d\mu(h)\, \hat{\mu}(p\gamma).\end{aligned} \tag{15}$$

But, by assumption, $\hat{\mu}(p\gamma) = \phi(p^2\gamma)$. Then using this and (15) we can write $\hat{\nu}(\gamma) = \phi(p^2\gamma) = \phi(p\gamma) = \phi(\gamma)$ since γ is in K.

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