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PFAFFIAN SYSTEMS FOR THE INFINITESIMAL AUTOMORPHISMS AND AN APPLICATION TO SOME DEGENERATE *CR* MANIFOLDS OF DIMENSION 3

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1. Introduction

In this paper we introduce a method of local construction of Pfaffian systems from given system of equations for infinitesimal automorphisms and apply the method to some degenerate CR structures.

Any system of partial differential equations can be expressed as an exterior differential system. For example, given a differential equation

$$F(x, y, u, u_x, u_y)=0, \text{ where } u_x=\frac{\partial u}{\partial x} \text{ etc.,} \tag{1}$$

we introduce new variables p and q for those partial derivatives. Then the equation is equivalent to the following exterior differential system in $\mathbf{R}^5$ consisting of a 0-form and a 1-form:

$$\begin{aligned} &F(x, y, u, p, q)=0 \text{ and} \\ &du-p\,dx-q\,dy=0, \end{aligned} \tag{2}$$

and to solve (1) is equivalent to finding an integral manifold of (2) on which

$$dx\wedge dy\neq 0 \text{ (cf. [7]).} \tag{3}$$

The theory of exterior differential systems was developed by E. Cartan. In real analytic case, he found a sufficient condition for the existence of an integral manifold which is tangent to a given integral element. This was later generalized to be the Cartan-Kähler theorem, which gives a construction of an integral manifold by a repeated application of the Cauchy-Kowalewski theorem (cf. [8], p. 174). For an exposition of this subject see [1]. We are concerned with a special case where a Pfaffian system (i. e. a system of differential 1-forms) can be constructed by expressing all the partial derivatives of a certain order,

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say m, of unknown functions as smooth functions of the derivatives of order $<m$. For example, suppose in (1) that we could find all the derivatives of order 3 of u as C^∞ functions of the derivatives of order <3 by successive differentiation of (1). We introduce new variables p, q, r, s and t for the partial derivatives u_x, u_y, u_{xx}, u_{xy} and u_{uu}, respectively. Then the following Pfaffian system can be constructed in $\mathbf{R}^8$:

$$
\begin{aligned}
(4) \qquad & du-p\,dx-q\,dy=0,\\
& dp-r\,dx-s\,dy=0,\\
& dp-s\,dx-t\,dy=0,\\
& dr-a_1dx-a_2dy=0,\\
& ds-a_3dx-a_4dy=0 \text{ and}\\
& dt-a_5dx-a_6dy=0, \text{ where each } a_i \ (i=1, \dots, 6)
\end{aligned}
$$

is a C^∞ functions of the variables (x, y, u, p, q, r, s, t). Then if $u=u(x, y)$ is a solution of (1) the submanifold

$$(x, y) \rightarrow (x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yy}(x, y))$$

is an integral manifold of (4). This is a situation much better than (2) because the Frobenius theorem gives both the existence and regularity, namely, if each a_i is C^∞ (resp. C^ω) then the integral manifold is C^∞ (resp. C^ω). The procedure of constructing new equations by differentiating the given differential equations and by introducing new variables as in construction of (4) is the so called method of prolongation which was also developed by Cartan. Two examples of computations of this scheme will be presented in the sequel.

2. Infinitesimal isometries and infinitesimal *CR* automorphisms

Let M be a differentiable manifold and G be a group of differentiable transformations of M. A vector field X on M is called an infinitesimal G-automorphism if the 1-parameter group $\{X_t\} \subset G$. Often the set of all infinitesimal automorphism forms a Lie algebra under the bracket of vector fields and we are interested in the dimension of this Lie algebra in connection with the following:

THEOREM 1 ([5], p. 13) *If the set $\{X\}$ of all infinitesimal G-automorphisms forms a finite dimensional Lie algebra, then G admits a Lie* *group structure whose associated Lie algebra is isomorphic to* $\{X\}$.

A best known example is that M is a Riemannian manifold with the Riemannian metric g and G is the group of all isometries of M onto itself. In this case, X is an infinitesimal isometry (also called a Killing vector field). Then the equation for X is

(5) $\quad L_X g=0$, where L is the Lie derivative (cf. [6]).

Equation (5) is linear in X. Furthermore, if X_1 and X_2 satisfy (5) then so does $[X_1, X_2]$, for

$$\begin{aligned} L_{[X_1, X_2]}g &= L_{X_1}L_{X_2}g - L_{X_2}L_{X_1}g \\ &= 0. \end{aligned}$$

Thus the infinitesimal isometries form a Lie algebra. We will show that this Lie algebra is finite dimensional by constructing (locally) a Pfaffian system of which X is a solution: Let $0 \in M$ and $\{e_1, \ldots\ldots, e_n\}$, $n=\dim M$, be an orthonormal frame on a neighborhood of 0. Then (5) is equivalent to

(5′) $\quad g(L_X e_i, e_j)+g(e_i, L_X e_j)=0$, for each $i, j=1, \ldots\ldots, n$.

Set $X=\sum_{s=1}^{n} f_s e_s$. Then

$$L_X e_k = \sum_{s=1}^{n} \{-(e_k f_s)e_s + f_s[e_s, e_k]\}.$$

Substituting in (5′) we get

(5″) $\qquad e_i f_j + e_j f_i + \langle \text{zeroth order } f's \rangle = 0,$

where $\langle\ \ \rangle$ denotes a linear combination with C^∞ coefficients. Apply e_k to (5″) to get

(6) $\quad e_k e_i f_j + e_k e_j f_i = \langle \text{derivatives of } f's \text{ of order} \leq 1 \rangle$.

Similarly,

(7) $\quad e_i e_j f_k + e_i e_k f_j = \langle \qquad \text{〃} \qquad \rangle$, and

(8) $\quad e_j e_k f_i + e_j e_i f_k = \langle \qquad \text{〃} \qquad \rangle$.

Consider (6)+(7)+(8). Since

$e_k e_i f_j = e_i e_k f_j - [e_i, e_k] f_j$ etc., LHS becomes
$2(e_i e_j f_k + e_j e_k f_i + e_k e_i f_j) + \langle \text{first order of } f's \rangle$.

But the sum of the first two terms is

$$\begin{aligned}&2(e_je_if_k+e_je_kf_i+\langle\text{first order of } f's\rangle)\\&=2e_j(e_if_k+e_kf_i)+\langle\text{first order of } f's\rangle\\&=\langle\text{derivatives of } f's \text{ of order}\leq 1\rangle \text{ by } (5'').\end{aligned}$$

Thus we get

$$e_ke_if_j=\langle\text{order}\leq 1 \text{ of } f's\rangle, \quad \text{for any} i, j, k=1\ldots\ldots, n,$$

from which a Pfaffian system as (4) can be constructed. If the Frobenius conditions are satisfied, there is a unique integral manifold passing through a given point $(0, f(0), p(0))$, where $f(0)$ has n components and p is the first order derivatives of f which has n^2 components. Therefore, the Lie algebra of the infinitesimal isometries has a dimension not exceeding $n+n^2$. Thus we proved the following theorem whose statement is well known in one way or another (cf. [5]).

THEOREM 2. *Let M be a C^∞ Riemannian manifold of dimension n. Then the set of all the infinitesimal isometries of M forms a Lie algebra of dimension not exceeding $n+n^2$. Furthermore, if M is C^ω, then a twice differentiable (C^2) infinitesimal isometry is C^ω.*

As for CR structures, there are many elegant ways of defining them, for example see [3], p. 48. However, we will define as follows: A CR (or Cauchy-Riemann) structure of a differentiable manifold M of dimension n is a pair $(H(M), J)$, where $H(M)$ is a sub-bundle of real codimension 1 of the tangent bundle $T(M)$ and $J: H(M)\to H(M)$ is a bundle isomorphism which maps each fibre onto itself and satisfies

$$J^2=-\text{Idendity}.$$

Then the complexification

$$H_{\mathbf{C}}(M)\equiv H(M)\otimes_{\mathbf{R}}\boldsymbol{C}$$

decomposes into a direct sum

$$H^{1,0}(M)+H^{0,1}(M),$$

where $H^{1,0}(M)$ (resp. $H^{0,1}(M)$) is the eigenspace of J associated to the eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$). A CR structure $(H(M), J)$ is said to be integrable if for any local sections Z and Z' of $H^{1,0}(M)$, their bracket $[Z, Z']$ is again contained in $H^{1,0}(M)$ on their common domain. A real hypersurface in a complex manifold has a natural integrable CR structure. A manifold with an integrable CR structure is called a CR manifold.

Let M be a CR manifold and $0\in M$. It is easy to show that there exists on a neighborhood of 0 a set of independent local sections of $H(M)$ of the form

$$\{Y_1, JY_1, \ldots, Y_m, JY_m\}, \quad (2m=n-1).$$

Thus n is odd. For each $i=1, \ldots, m$ let

$$Z_i = Y_i - \sqrt{-1}JY_i \text{ and}$$
$$\bar{Z}_i = Y_i + \sqrt{-1}JY_i.$$

Then we see that $\{Z_i\}_{i=1,\ldots,m}$ (resp. $\{\bar{Z}_i\}_{i=1,\ldots,m}$) is a local basis of $H^{1,0}(M)$ (resp. $H^{0,1}(M)$). Therefore, $\overline{H^{1,0}}(M)=H^{0,1}(M)$. Now choose on a neighborhood of 0 a smooth nonvanishing real vector field T which is transversal to $H(M)$. For each $i, j=1, \ldots, m$, let $\rho_{i\bar{j}}$ be a function defined by

$$[Z_i, \bar{Z}_j] = \sqrt{-1}\rho_{i\bar{j}}T \pmod{H_C(M)}.$$

Then the $m\times m$ matrix $(\rho_{i\bar{j}})$ is hermitian, and called the Levi form. A CR structure is said to be nondegenerate (resp. degenerate) at 0 if det $(\rho_{i\bar{j}})$ is nonzero (resp. zero) at 0. This definition is in dependent of choice of the basis $\{Z_i\}$ and T. A diffeomorphism ϕ of a CR manifold M onto another M' is called a CR equivalence if it preserves the CR structure, namely

$$d\phi : H(M) \to H(M') \text{ and}$$
$$d\phi \circ J = J \circ d\phi \quad \text{on } H(M).$$

A vector field X on a CR manifold M is an infinitesimal CR automorphism if and only if for any local section Y of $H(M)$ $L_X Y$ is again a section of $H(M)$ and

$$L_X(JY) = J(L_X Y). \tag{9}$$

We will construct a Pfaffian system for X by differentiating the equation (9). An important difference between (5) and (9) is that in (5) all the directional derivatives are equally involved while (9) has control in $H(M)$ only and misses one direction that is perpendicular to $H(M)$. Roughly speaking, the derivatives in this missing direction can be obtained by considering $L_x[Y, JY]$, which will be made clear in the sequel.

THEOREM 3 (E. Cartan). *Let M be a C^∞ hypersurface in $\boldsymbol{C}^2$ with a nondegenerate Levi form. Then the set of all the smooth infinitesimal CR automorphisms of M forms a finite dimensional Lie algebra.*

We will state a generalization of Theorem 3. First, express (9) in complex notations: Let Y be a local section of $H(M)$ over a neighborhood of $0\in M$ and let

$$
\begin{aligned}
Z &\equiv Y-\sqrt{-1}\,JY. \quad \text{Then}\\
[X, Z] &\equiv L_X Z\\
&\equiv L_X Y-\sqrt{-1}\,L_X(JY)\\
&= L_X Y-\sqrt{-1}\,J(L_X Y), \quad \text{by (9)},
\end{aligned}
$$

which is a section of $H^{1,0}(M)$, this we have

(10) $[X, Z]=\alpha Z$, for some function α on a neighborhood of 0.

Our main theorem is the following

THEOREM 4. *Let M be a C^∞ hypersurface in $\boldsymbol{C}^2$. Let $0\in M$ and Z be a C^∞ local section of $H^{1,0}(M)$ over a neighborhood of 0. If there exists an integer k such that the k-th bracket*

$$
[\cdots[[Z, \underbrace{\bar{Z}], \bar{Z}], \cdots, \bar{Z}}_{k\ times}]\neq 0, \quad mod\ [Z, \bar{Z}] \tag{11}
$$

then the set of all the C^∞ infinitesimal CR automorphisms of M forms a finite dimensional Lie algebra. Furthermore, if M is C^ω then any C^{2k+1} infinitesimal CR automorphism is C^ω.

Observe that the condition (11) is independent of choice of Z and the special case of $k=1$ is the Theorem 3. In the next section we will construct a Pfaffian system for which X is a solution. Then the same argument as in the proof of Theorem 2 proves Theorem 4.

3. Proof of the Theorem 4

Equation (10) is linear in X. Furthermore, if X_1 and X_2 satisfy (10), so does $[X_1, X_2]$ (use the Jacobi identity). Therefore, infinitesimal CR automorphisms form a Lie algebra. Let k be the first integer for which (11) holds. Choose a nonvanishing real vector field T which is transversal to $H(M)$ on a neighborhood of 0. We fix an open set U=the common domain of Z and T. Let

(12) $\quad [Z, \bar{Z}]=\sqrt{-1}\,\rho T+aZ-\bar{a}\bar{Z}$ and

(13) $\quad [T, Z]=b_1T+b_2Z+b_3\bar{Z}$, or equivalently
$[T, \bar{Z}]=\bar{b}_1T+\bar{b}_3Z+\bar{b}_2\bar{Z}$.

Substitute (12) in (11), to get
$[\cdots[[Z, \bar{Z}], \bar{Z}], \cdots, \bar{Z}]=\{(-1)^{k-1}\sqrt{-1}\,\bar{Z}^{k-1}\rho+\gamma\}\,T, \mod(Z, \bar{Z})$,
where γ is an element of the algebra generated by $\rho, \bar{Z}\rho, \cdots, \bar{Z}^{k-2}\rho$ over the ring of C^∞ functions on U. Thus the condition (11) is equivalent to

$$\bar{Z}^{k-1}\rho(0)\neq 0 \text{ and } \bar{Z}^s(0)=0, \quad \forall s<k-1.$$

We set

(14) $\quad X=fZ+\bar{f}\bar{Z}+gT$, where f is complex valued and g is real.

We will express all the partial derivatives of f and g of order $2k+1$ as linear combinations with C^∞ coefficients of derivatives of f and g of order $\leq 2k$. Substitute (14) in (10) to get

$$\alpha Z=-(Zf)Z-(Z\bar{f})\bar{Z}-(Zg)T-\bar{f}[Z, \bar{Z}]+g[T, Z]$$
substitute (12) and (13) for $[Z, \bar{Z}]$ and $[T, \bar{Z}]$, resp.
$$=\{-Zf-a\bar{f}+b_2g\}Z+\{-Z\bar{f}+\bar{a}\bar{f}+b_3g\}\bar{Z}$$
$$+\{-Zg-\sqrt{-1}\,\rho\bar{f}+b_1g\}\,T.$$

By comparing components of two sides we get

(15) $\quad \alpha=-Zf-a\bar{f}+b_2g$

(16) $\quad Z\bar{f}=\bar{a}\bar{f}+b_3g$, or equivalently
$\bar{Z}f=af+\bar{b}_3g$, and

(17) $\quad Zg=-\sqrt{-1}\,\rho\bar{f}+b_1g$, or equivalently
$\bar{Z}g=\sqrt{-1}\,\rho f+\bar{b}_1g$

Now consider $[X, [Z, \bar{Z}]]$, which is equivalent to considering $L_X([Y, JY])$ mentioned earlier. By Jacobi identity

$$\begin{aligned}[X, [Z, \bar{Z}]]&=-[Z, [\bar{Z}, X]]-[\bar{Z}, [X, Z]]\\ &=[Z, \bar{\alpha}\bar{Z}]+[\alpha Z, \bar{Z}], \text{ by (10)}\\ &=(\alpha+\bar{\alpha})[Z, \bar{Z}] \mod (Z, \bar{Z})\end{aligned}$$

$$=\sqrt{-1}\,\rho(-Zf-\bar{Z}\bar{f})+\langle \rho f, \rho\bar{f}, \rho g\rangle \bmod (Z,\bar{Z}) \text{ by (15).}$$

On the other hand

$$\begin{aligned}[X,[Z,\bar{Z}]]&=[X,\sqrt{-1}\,\rho T+aZ-\bar{a}\bar{Z}]\\ &=\sqrt{-1}(X\rho)T+\sqrt{-1}\,\rho[X,T] \bmod (Z,\bar{Z}),\\ &\quad\text{substitute (14) for } X\\ &=\{-\sqrt{-1}\rho\, Tg+\langle f,g,\bar{f}\rangle\}\,T \bmod (Z,\bar{Z}).\end{aligned}$$

By equating the above two we get

$$-\rho Tg+\rho Zf+\rho\bar{Z}\bar{f}=\langle f,g,\bar{f}\rangle. \tag{18}$$

Now we apply $\bar{Z}$ repeatedly to (18). Observe first that

$$\begin{aligned}&(19)\quad \bar{Z}Tg=T\bar{Z}g-[T,\bar{Z}]g\\ &\qquad\qquad \text{substitute (17) and (13)}\\ &\qquad\quad =\sqrt{-1}\,\rho Tf+\langle f,g,\bar{f},Tg\rangle, \text{ or equivalently}\\ &\qquad ZTg=-\sqrt{-1}\,\rho T\bar{f}+\langle f,g,\bar{f},Tg\rangle. \text{ Similarly}\\ &(20)\quad \bar{Z}Zf=-\sqrt{-1}\,\rho Tf+\langle f,g,\bar{f}\rangle \text{ by (16) and (12), or equivalently}\\ &\qquad Z\bar{Z}\bar{f}=\sqrt{-1}\,\rho T\bar{f}+\langle f,g,\bar{f}\rangle \text{ and}\\ &(21)\quad \bar{Z}Tf=\langle Tf,Tg,Zf,f,g,\bar{f}\rangle \text{ by (16) and (13), or equivalently}\\ &\qquad ZT\bar{f}=\langle T\bar{f},Tg,\bar{Z}\bar{f},f,\bar{f},g\rangle.\end{aligned}$$

By induction, we have for any positive integer q,

$$\begin{aligned}&(19')\quad \bar{Z}^qTg=(\sqrt{-1}\bar{Z}^{q-1}\rho+\gamma_1)Tf+\langle f,g,\bar{Z}^s\bar{f}(s\le q),Tg\rangle\\ &(20')\quad \bar{Z}^qZf=(-\sqrt{-1}\bar{Z}^{q-1}\rho+\gamma_2)Tf+\langle f,g,\bar{Z}^s\bar{f}(s\le q),Tg\rangle \text{ and}\\ &(21')\quad \bar{Z}^qTf=\langle Tf,Tg,f,g,\bar{Z}^s\bar{f}(s\le q)\rangle, \textit{ where } \gamma_1 \textit{ and } \gamma_2\\ &\qquad \text{are elements of the algebra generated by } \rho,\bar{Z}\rho,\dots,\ \bar{Z}^{q-2}\rho.\end{aligned}$$

By (19)–(21), $\bar{Z}$ applied to (18) becomes

$$-(\bar{Z}\rho)Tg-2\sqrt{-1}\,\rho^2Tf+(\bar{Z}\rho)Zf=\langle\rho Tg,f,g,\bar{Z}^2f,\bar{Z}\bar{f},\bar{f}\rangle.$$

Now for any integer p we denote by δ_p a finite sum of products of $\{\bar{Z}^s\rho; s=0,1,2,\dots\}$ such that

$$\begin{aligned}\bar{Z}^t\delta_p(0)&\neq 0 \quad \text{if } t=p\\ &=0 \quad \text{if } t<p.\end{aligned}$$

Let q be any positive integer. Apply $\bar{Z}^q$ to (18). Then by induction on q and by (19′)–(21′) we have

$$\delta_{2k-1-q}Tf+\delta_{k-1-q}Zf+\delta_{k-1-q}Tg=\langle f,g,\bar{Z}^s\bar{f}(s\le q+1)\rangle \tag{22}$$

Then (22) with $q=k-1$ is

(23) $\delta_k Tf+\delta_0 Zf+\delta_0 Tg=\langle f, g, \bar{Z}^s\bar{f}(s\leq k)\rangle$ and (22) with $q=2k-1$ is

(24) $\delta_0 Tf+\delta_{-k}Zf+\delta_{-k}Tg=\langle f, g, \bar{Z}^s\bar{f}(s\leq 2k)\rangle$.

Solve (23) and (24) simultaneously for Tf and Zf to get

$$Tf=\langle Tg, \bar{f}, g, \bar{Z}^s\bar{f}(s\leq 2k)\rangle \text{ and}$$
$$Zf=\langle \qquad \text{''} \qquad \rangle, \text{ or equivalently.}$$

(25) $$\begin{cases} T\bar{f}=\langle Tg, f, g, Z^s f(s\leq 2k)\rangle \text{ and} \\ \bar{Z}\bar{f}=\langle \qquad \text{''} \qquad \rangle. \end{cases}$$

Now for any positive integer q, apply $\bar{Z}^q$ to (25). Then by (19′)–(21′)

(26) $$\begin{cases} T\bar{Z}^q\bar{f}=\langle Tg, T^tZ^sf(t+s\leq 2k), T\bar{Z}^s\bar{f}(s<q), \bar{Z}^s\bar{f}(s\leq q)\rangle \\ \text{and} \\ \bar{Z}^{q+1}\bar{f}=\langle \qquad \text{''} \qquad \rangle. \end{cases}$$

In order to express all the partial derivatives of f and g of order $2k+1$ in terms of lower order terms, first observe that mixed $(Z, \bar{Z})$ derivatives of f of order $2k+1$ always reduce to lower order by (16) and $(Z, \bar{Z})$ derivatives of g always reduce to lower order by (17). Therefore, it suffices to express the derivatives

$T^tZ^sf(t+s=2k+1)$ and $T^{2k+1}g$ as linear combinations of derivatives of order $\leq 2k$.

Apply Z^{2k} to (18). Then the first term becomes by (19) $\langle T\bar{f}, f, \bar{f}, Tg\rangle$, the second term becomes $\rho Z^{2k+1}f+\langle Z^sf(s\leq 2k)\rangle$ and the third term becomes by (19)–(21)

$\langle T\bar{f}, Tg, Z\bar{f}, f, \bar{f}, g\rangle$. Thus we get

(27) $$\rho Z^{2k+1}f+\langle Z^sf(s\leq 2k)\rangle=\langle T\bar{f}, Tg, \bar{Z}\bar{f}, f, \bar{f}, g\rangle.$$

Now we apply $\bar{Z}$ repeatedly to (27). The second term does not raise its order as we apply $\bar{Z}$ repeatedly by (16) and the RHS becomes

$$\langle T\bar{Z}^s\bar{f}, \bar{Z}^s\bar{f}(s=1, 2, \ldots), \; T\bar{f}, Tg, f, \bar{f}, g\rangle.$$

Consider the first term.

$$\bar{Z}(\rho Z^{2k+1}f)=\bar{Z}\rho Z^{2k+1}f+\rho\bar{Z}Z^{2k+1}f$$
but $\bar{Z}Z^{2k+1}f=-(2k+1)\sqrt{-1}\,\rho TZ^{2k}f+\langle TZ^sf(s\leq 2k-1),$
$Z^sf(s\leq 2k), g\rangle$, therefore

$=\bar{Z}\rho Z^{2k+1}f-(2k+1)\sqrt{-1}\,\rho^2 TZ^{2k}f+\langle$derivatives of order $\leq 2k\rangle$.

Therefore, $\bar{Z}$ applied to (18) gives

$$\delta_{k-2}Z^{2k+1}f-\delta_{2(k-1)}TZ^{2k}f+\langle\text{derivatives of } f \text{ or order}\leq 2k\rangle$$
$$=\langle T\bar{Z}\bar{f},\ T\bar{f},\ Tg,\ \bar{Z}^2\bar{f},\ \bar{f}\bar{Z},\ \bar{f},\ g\rangle.$$

By induction we have for any positive integer q

$$\begin{aligned}(28)\quad & \bar{Z}^q\delta_{k-1}Z^{2k+1}f-\bar{Z}^{q-1}\delta_{2(k-1)}TZ^{2k}f+\bar{Z}^{q-2}\delta_{3(k-1)}T^2Z^{2k-1}f-\cdots\\ &+(-1)^t\bar{Z}^{q-t}\delta_{(t+1)(k-1)}T^tZ^{2k+1-t}f+\cdots\\ &+(-1)^{2k+1}\bar{Z}^{q-(2k+1)}\delta_{(2k+2)(k-1)}T^{2k+1}f\\ &+\langle\text{derivatives of } f \text{ of order}\leq 2k\rangle\\ &=\langle T\bar{Z}^s\bar{f}(s\leq q),\ \bar{Z}^s\bar{f}(s\leq q+1),\ Tg,\ g,\ \bar{f}\rangle.\end{aligned}$$

Consider (28) with $q=k-1,\ 2k-1, \ldots,\ (2k+2)k-1$. The highest order terms of LHS of (28) are

$$\begin{aligned}q=k-1:&\ \delta_0Z^{2k+1}f-\delta_kTZ^{2k}f+\delta_{2k}T^2Z^{2k-1}f-\cdots\\ q=2k-1:&\ {}^*Z^{2k+1}f-\delta_0TZ^{2k}f+\delta_kT^2Z^{2k-1}f-\cdots\\ &\ \vdots\end{aligned}$$

Nonzero coefficients lie in below the diagonal and the diagonal elements are $\pm\delta_0$. Thus we can solve the simultaneous equations (28) with $q=k-1,\ 2k-1, \cdots, (2k+2)k-1$ for $Z^{2k+1}f,\ TZ^{2k}f, \cdots, T^{2k+1}f$. But the RHS of (28) always reduces to terms of order$\leq 2k$ by (26). Thus we get $T^tZ^sf(t+s=2k+1)$ as linear combination of derivatives of order$\leq 2k$.

To get $T^{2k+1}g$, apply T^{2k} to (18). Then

$$\rho T^{2k+1}g=\langle T^tZf,\ T^tf,\ T^tg,\ T^t\bar{f}(t\leq 2k)\rangle.$$

Apply $\bar{Z}^{k-1}$ to the above. Then the RHS always reduces to the order $\leq 2k$ by the previous step and the LHS becomes $(Z^{k-1}\rho+\gamma)T^{2k+1}g+\langle T^tZ^sf(t+s\leq 2k+1),\ T^tg(t\leq 2k)\rangle$ by (16) and (17), where γ is an element of the algebra generated by $\{Z^s\rho;\ s\leq k-2\}$, therefore, $\gamma(0)=0$. Thus we get

$$T^{2k+1}g=\text{linear combination of derivatives of order} \leq 2k.$$

This completes the proof. Observe that this proof is still valid after the hypersurface M in Theorem 4 is replaced by an abstract CR manifold.

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