J. Korean Math. Soc. **21** (1984), *No.* 1, 1–7

EXTENSIONS OF THE WEAK CONTRACTIONS OF DUGUNDJI AND GRANAS

SEHIE PARK AND WON KYU KIM

1. Introduction

A Banach contraction is a selfmap f of a metric space (X, d) satisfying $d(fx, fy) \leq \alpha\, d(x, y)$ for all $x, y \in X$ and for some $\alpha \in [0, 1)$. The well-known Banach contraction principle states that for complete X, such f has a unique fixed point p and $f^n x \to p$ for all $x \in X$. There have been numerous literatures on extensions of the principle.

In [3], J. Dugundji and A. Granas extend the principle to a contractive type map which is called a weak contraction and obtain some applications including a domain invariance theorem. They also introduce the concept of weakly expansive maps and establish some of their properties.

In the present paper, we show that the Dugundji-Granas contraction is actually particular to some well known other contractive type maps and that their fixed point results are actually particular to those in [6], [7], [8], and [9]. Moreover, following the methods in [7], [8], [9], we show that fixed point theorems on weak contractions and on weakly expansive maps can be unified to a single theorem. Furthermore, we extend the domain invariance theorem for weakly contractive fields to the one for Meir-Keeler type contractive fields. In the sequel, we follow the notations of [7].

2. Comparisons of contractive type conditions

Let R_+ denote the set of nonnegative reals. A map $\phi : R_+ \to R_+$ is said to be compactly positive if $\inf\{\phi(t) \mid a \leq t \leq b\} = \lambda(a, b) > 0$ for any $b > a > 0$ [3].

Consider the following conditions on a selfmap f of a metric space

Received August 25, 1983.

*A research supported by a grant from the Korea Science and Engineering Foundation in 1983-84.

(X, d):

(Dd) There exists an increasing right continuous function $\phi : R_+ \to R_+$ such that $\phi(t) < t$ for $t > 0$ and, for any $x, y \in X$, we have

$$d(fx, fy) \leq \phi(d(x, y)).$$

(DG) There exists a compactly positive function $\phi : R_+ \to R_+$ such that for any $x, y \in X$,

$$d(fx, fy) \leq d(x, y) - \phi(d(x, y)).$$

(Cd) Given $\varepsilon > 0$, there exist $\varepsilon_0 < \varepsilon$ and $\delta_0 > 0$ such that for any $x, y \in X$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta_0 \quad \text{implies} \quad d(fx, fy) \leq \varepsilon_0.$$

(Bd) Given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in X$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(fx, fy) < \varepsilon.$$

Note that the condition (Dd) is due to Browder [2], (DG) to Dugundji–Granas [3], (Cd) to Boyd–Wong [1] (also see Hegedüs–Szilagyi [4]), and (Bd) to Meir–Keeler [6].

The following is basic.

LEMMA. (Dd) $\Longrightarrow$ (DG) $\Longrightarrow$ (Cd) $\Longrightarrow$ (Bd).

Proof. (Dd) $\Longrightarrow$ (DG) is given as Proposition (3.2) by Dugundji–Granas [3]. (Cd) $\Longrightarrow$ (Bd) is given by Meir–Keeler [6]. It remains to show that (DG) $\Longrightarrow$ (Cd). In [3], it is shown that (DG) is equivalent to the following condition of Krasnoselskij [5]:

(1) There exists a map $\alpha : R_+ \to R_+$ satisfying $\sup\{\alpha(t) \mid a \leq t \leq b\} < 1$ for any $b > a > 0$, such that for any $x, y \in X$,

$$d(fx, fy) \leq \alpha(d(x, y)) \cdot d(x, y).$$

On the other hand, in [4], it is shown that (Cd) is equivalent to the following:

(2) There exists a map $\alpha : R_+ \to [0, 1)$ such that for any $\varepsilon > 0$ there exists a $\delta > 0$ with $\sup\{\alpha(t) \mid \varepsilon \leq t < \varepsilon + \delta\} < 1$ and, for any $x, y \in X$,

$$d(fx, fy) \leq \alpha(d(x, y)) \cdot d(x, y).$$

Then it is clear that (1) $\Longrightarrow$ (2). This completes our proof.

REMARK. An example showing (Bd)$\not\Longrightarrow$(Cd) is given by Meir-Keeler [6].

3. Fixed point theorems

Let f be a continuous selfmap of a metric space (X, d), $C_f = \{g : X \to X \mid fg = gf,\ gX \subset fX\}$. For $x_0 \in X$ the sequence $\{fx_n\}_{n=1}^{\infty}$ is called the f-iteration of x_0 under g, as defined by $fx_n = gx_{n-1}$, $n = 0, 1, 2, \dots$, with the understanding that, if $fx_n = fx_{n+1}$ for some n, then $fx_{n+j} = fx_n$ for each $j \geq 0$. The set $\{fx_n\}_{n=1}^{\infty}$ will be denoted by $O(x_0)$. A point $x_0 \in X$ is said to be regular if $\operatorname{diam} O(x_0) < \infty$.

The following is a consequence of Theorem 2(Cδ)$'$ in [7].

THEOREM 3. 1. *Let f be a continuous selfmap of a complete metric space (X, d) and $g \in C_f$ continuous. Suppose that X contains a regular point and that*

(Cδ)$'$ *for any $\varepsilon > 0$, there exist $\varepsilon_0 < \varepsilon$ and $\delta_0 > 0$ such that for any regular points $x, y \in X$,*

$$\varepsilon \leq \operatorname{diam}\ (O(x) \cup O(y)) < \varepsilon + \delta_0 \text{ implies } d(gx, gy) \leq \varepsilon_0.$$

Then f and g have a unique common fixed point p in X, and, for any regular $x_0 \in X$, any f-iteration of x_0 under g converges to some $\xi \in X$ satisfying $f\xi = p$.

THEOREM 3. 2. [9, Theorem 4] *Let f be a continuous selfmap of a complete metric space (X, d), $g \in C_f$ continuous and f, g satisfying the following condition:*

(Bk)$'$ *For each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varepsilon \leq \max\{d(fx, fy),\ d(fx, gx),\ d(fy, gy),\ [d(fx, gy) + d(fy, gx)]/2\} < \varepsilon + \delta$ implies $d(gx, gy) < \varepsilon$.*

Then f and g have a unique common fixed point p in X, and, for any $x_0 \in X$, any f-iteration of x_0 under g converges to some $\xi \in X$ satisfying $f\xi = p$.

If $f = 1_X$, the condition (Bk)$'$ will be denoted by (Bk) [10]. Now, from Theorem 3. 2, we have

THEOREM 3. 3. [8, Theorem 2. 4] *Let f be a continuous selfmap of a complete metric space X, and $g \in C_f$, satisfying*

(Bd)′ *for each* $\varepsilon>0$ *there exists a* $\delta>0$ *such that, for all* $x, y\in X$,

$$\varepsilon \le d(fx,fy) < \varepsilon+\delta \text{ implies } d(gx,gy) < \varepsilon.$$

Then f and g have a unique common fixed point p in X, and, for any $x_0\in X$, *any f-iteration of* x_0 *under g converges to some* $\xi\in X$ *satisfying* $f\xi=p$.

Now consider the following condition on f and g:

(DG)′ There exists a compactly positive function $\phi : R_+ \to R_+$ such that for any $x, y \in X$, $d(gx,gy) \le d(fx,fy) - \phi(d(fx,fy))$.

Imitating (DG)$\Longrightarrow$(Bd)$\Longrightarrow$(Bk) and (DG)$\Longrightarrow$(Cd)$\Longrightarrow$(Cδ) as in Section 2 and in [4], [10], we have (DG)′$\Longrightarrow$(Bk)′ and (DG)′ $\Longrightarrow$(Cδ)′.

Therefore, from Theorems 3.1 and 3.3 we obtain the following

THEOREM 3.4. *Let f be a continuous selfmap of a complete metric space X and* $g\in C_f$ *satisfying the condition* (DG)′. *Then f and g have a unique common fixed point p in X, and, for any* $x_0 \in X$, *any f-iteration of* x_0 *under g converges to some* $\xi\in X$ *satisfying* $f\xi=p$.

Theorem 3.4 unifies the main fixed point results on weak contractions and on weak expansions in [3]. In fact, by putting $f=1_X$, we obtain

COROLLARY 1 [3, Theorem (1.4)] *Let* (X,d) *be complete and* $g : X\to X$ *a weak contraction, that is, g satisfies* (DG). *Then g has a fixed point p, and* $g^n x\to p$ *for each* $x\in X$.

By putting $g=1_X$ in Theorem 3.4, we have

COROLLARY 2 [3, Remark in Section 5] *Let* (X,d) *be complete and* $f : X\to X$ *be a surjective weak expansion, that is, there exists a compactly positive function* $\phi : R_+ \to R_+$ *such that for any* $x, y \in X$,

$$d(fx,fy)\ge d(x,y) + \phi(d(fx,fy)).$$

Then f has a fixed point p, and, $f^n x\to p$ *for each* $x\in X$.

4. Domain invariance for the Meir-Keeler type contractive fields

Let E be a Banach space and $U\subset E$ open. Given $F : U\to E$, the map $f : U\to E$ given by $fx = x - Fx$ is called the field (of displacements) associated with F.

LEMMA. *Let $F : E \to E$ be a selfmap of a metric space (E, d) satisfying* (Bd). *For any $r > 0$, if $d(x, Fx) \leq \delta(r)$ for some $x \in E$, then F maps $B(x, r+\delta(r))$ into itself.*

Proof. Suppose $y \in B(x, r+\delta(r))$. If $d(x, y) < r$, then $d(Fx, Fy) < r$, since F is contractive. If $r \leq d(x, y) < r+\delta(r)$, then $d(Fx, Fy) < r$, since F satisfies (Bd). Therefore, in any case,

$$d(x, Fy) \leq d(x, Fx)+d(Fx, Fy) < \delta(r)+r.$$

REMARK. Actually F maps $\overline{B}(x, r+\delta(r))$ into itself since F is continuous.

THEOREM 4. 1. *Let E be a Banach space, $U \subset E$ open, $F : U \to E$ satisfy* (Bd), *and $f : U \to E$ its associated field. Then*

(a) *$f : U \to E$ is an open map, and*
(b) *$f : U \to fU$ is a homeomorphism.*

Proof. (a) For each $x \in U$ and a ball $B(x, r) \subset U$, we show that there is a ball $B(fx, \rho) \subset f(B(x, r))$. Suppose $0 < r' < r$. Then there exists $\delta(r') > 0$ satisfying (Bd) with respect to F.

Case (i) $r'+\delta(r') < r$: Choose any $x_0 \in B(fx, \delta(r'))$. Define $G : \overline{B}(x, r') \to E$ by $Gy = x_0 + Fy$. Then G also satisfies (Bd). Since

$$\|Gx-x\| = \|x_0+Fx-x\| = \|x_0-fx\| < \delta(r')$$

by Lemma, G maps $\overline{B}(x, r'+\delta(r'))$ into itself. Therefore, G has a unique fixed point y_0 in $\overline{B}(x, r'+\delta(r'))$. Since $y_0 = Gy_0 = x_0 + Fy_0$, we have $fy_0 = x_0$. Since x_0 is arbitrary in $B(fx, \delta(r'))$, we have

$$B(fx, \delta(r')) \subset f(\overline{B}(x, r'+\delta(r')) \subset f(B(x, r)).$$

Therefore f is open.

Case (ii) $r \leq r'+\delta(r')$: Let $\delta = (r-r')/2 > 0$. If $r' \leq \|x-y\| < r'+\delta$, then $\|Fx-Fy\| < r'$. As in the Case (i), we choose $x_0 \in B(fx, \delta)$ and define $G : \overline{B}(x, r') \to E$ by $Gy = x_0 + Fy$. Then G satisfies (Bd) and

$$\| Gx - x \| = \| x_0+Fx - x \| = \| x_0 - fx \| < \delta.$$

Hence, by Lemma, G maps $\overline{B}(x, r'+\delta)$ into itself. Therefore, we have

$$B(fx, \delta) \subset f(\overline{B}(x, r'+\delta)) \subset f(B(x, r)),$$

and f is open.

(b) Let $fx = fy$. Then $x - Fx = y - Fy$ implies $x - y = Fx - Fy$. Since F is contractive, we should have $x = y$. Therefore f is injective. Since f is a continuous open bijection between U and fU, f is a homeomorphism.

REMARK. Note that Dugundji and Granas [3, Theorem 2. 1] obtained Theorem 4. 1 with respect to the condition (DG) instead of (Bd).

Using the above theorem, we can obtain some corollaries corresponding to results of Dugundji–Granas [3].

COROLLARY 1. *Let E be a Banach space and $F : E \to E$ satisfy* (Bd). *Then the associated field f is a homeomorphism of E onto itself.*

COROLLARY 2. *Let X be any space, E a Banach space and $f : X \to E$ an embedding of X onto an open set $U \subset E$. Let $g : X \to E$ be a map such that $g \circ f^{-1} : U \to E$ satisfies* (Bd). *Then $x \to fx - gx$ is also an open embedding of X into E.*

COROLLARY 3. [3, Proposition 2. 4] *Let E be a Banach space, $U \subset E$ open, and $f : U \to E$ a C^1-map. If its derivative $Df(x_0) : E \to E$ is an isomorphism, then f maps a neighborhood of x_0 homeomorphically onto a neighborhood of fx_0.*

Let (E, d) be a metric space. By imitating the definition of a contractive type map, we define the following expansive type map.

DEFINITION. Let (E, d) be a complete metric space. A map $f : E \to E$ is called a (Bd)-type expansive map if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq d(fx, fy) < \varepsilon + \delta(\varepsilon) \text{ implies } d(x, y) < \varepsilon.$$

Of course, such a map need not be continuous. If f is a surjective (Bd)-type expansive map, then f^{-1} is well defined and satisfies (Bd). Therefore we obtain a result by using Theorem 4. 1.

THEOREM 4. 2. *Let E be a Banach space and $F : E \to E$ a* (Bd)-*type expansive surjective map. Then the associated field f is a bijective open map.*

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Seoul National University
Seoul 151, Korea