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A note on S-closed space

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S-closed 공간에 관하여

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요 약

위상 공간 X의 모든 Semi-open cover에 대하여 그들의 closure의 합이 X를 cover한 유한 부분 족이 존재할 때 위상 공간 X를 S-closed라고 한다. 이 논문에서는 S-closed와 semi-closed set 사이의 관계를 조사하였고 Haussdorff 공간과 S-closed 공간에서 extremally disconnected와 semi-continuous의 성질을 조사하였다.

Ⅰ. INTRODUCTION

The concept of an S-closed set using the properties of semi-open was defined. In this paper, a chacterization of S-closed spaces are given using a certain class of functions. Separation axioms are not used unless otherwise specified.

Ⅱ. DEFINITIONS

Def. 1) A Hausdorff space X is H-closed if and only if for every open cover $\{U_a | a \in \Lambda\}$, there exits a finite subfamily $\{U_{a_i} | i=1, 2, \cdots, n\}$ such that the union of there closures cover X.

Def. 2) A subset V of topological space is semi-open if and only if $V^{\circ} \subset V \subset \overline{V}$ (V° denotes interior of V and $\overline{V}$ denotes closure of V).

Def. 3) A topological space X is S-closed if and only if every semi-open cover of X has a finite subcollection whose closure cover X.

Def. 4) A function $f : X \rightarrow Y$ is said to be semi-continuous if and only if the inverse image of semi-open (open) set is semi-open.

Def. 5) A topological space is extremally disconnected if the closure of every open set is open.

Def. 6) If A is any subset of a topological space X, then the semi-closure of A is the intersection of all semi-closed sets in X that contain A. ($\underline{A}$ denotes semiclosure of A)

Ⅲ. THEOREMS

Theorem 1) For a topological space the fol-

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lowing are equivalent.

a) X is S-closed

b) For each family of semi-closed sets $\{Fa\}$ (i.e. each Fa is the complement of a semi-open set) such that $\cap Fa=\phi$, there exists a finite subfamily $\{Fa_i\}^n_{i=1}$ such that $\cap^n_{i=1}(Fa_i)^\circ=\phi$.

c) Each filterbase $F=\{Aa\}$ s-accumulates to some point $x_0 \in X$.

d) Each maximum filterbase F s-converges.

Proof) a)→d). Let $F=\{Aa\}$ be a maximum filterbase. Suppose that F does not s-converges to any point; therefore F does not s-accumulate to any point. Then for every $x \in X$, there exists a semi-open set $V(x)$ containing x and an $Aa_{(x)} \in F$ such that $Aa_{(x)} \cap \overline{V(x)}=\phi$. Obviously $\{V(x)|x \in X\}$ is a semi-open cover for X and there exsits a finte subfamily such that $\cap^n_{i=1}V(x_i)=X$. Since F is a filterbase, there exists an $A_0 \in F$ such that $A_0 \subset \cap^n_{i=1}Aa(x_i)$. Hence $A_0 \cap V(x_i)=\phi$, $1 \leq i \leq n$, which implies

$A_0 \cap (\cup^n_{i=1}V(x_i))=A_0 \cap X=\phi$,

contradicting the essential fact that $A_0 \neq \phi$.

d)→c). Each filterbase is contained in a maximal filter-base.

c)→b). Let $\{Fa\}$ be a collection of semi-closed sets such that $\cap Fa=\phi$. Suppose that for every finite subfamily, $\cap^n_{i=1}(Fa_i)^\circ \neq \phi$. Therefore $F=\{\cap^n_{i=1}(Fa_i)^\circ | n \in Z^+,\ Fa_i \in \{Fa\}\}$ forms a filterbase. From hypothesis, F s-accumulates to some point $x_0 \in X$. This implies that for every semi-open set $V(x_0)$ containing x_0, $Fa^\circ \cap \overline{V(x_0)} \neq \phi$, for every $a \in A$. Since $x_0 \notin \cap Fa$ there exists an $a_0 \in A$ such that $x_0 \notin Fa_0$. Hence x_0 is contained in the semi-open set $X-Fa_0$. Therefore

$(Fa_0)^\circ \cap \overline{(X-Fa_0)}=(Fa_0)^\circ \cap (X-(Fa_0)^\circ)=\phi$,

contradicting the fact that F s-accumulates to x_0.

b)→a). Let $\{V_a\}$ be a semi-open covering of X. Then $\cap(X-V_a)=\phi$. By hypothesis, there exists a finite subfamily such that $\cap^n_{i=1}(X-Va_i)^\circ=\cap^n_{i=1}(X-\overline{V}a_i)=\phi$. Therefore, $\cap^n_{i=1}\overline{V}a_i=X$, and consequently X is S-closed.

Theorem 2) If X is a S-closed regular space, then X is extremally disconnected.

Proof) Suppose that X is not extremally disconnected. Then there exists a regular open set $O \subset X$ such that $\bar{O}-O$ and $X-\bar{O}$ are nonempty. Let $x \in \bar{O}-O$. Then for every neighborhood V of x, $V \cap O \neq \phi$. Therefore, $F=\{(V \cap O)\}$ forms a filterbase in $\bar{O}$. Since $\bar{O}$ is S-closed, F s-accumulates to some point x_0 in $\bar{O}$. The filterbase also converges to x. We claim that $x_0 \notin \bar{O}-O$; for if it were, then $x_0 \in X-O$ and every member of F would have to intersect $X-O$, an impossibility. Thus, $x_0 \in O$. There exists an open set U such that $x_0 \in U \subset \bar{U} \subset O$ and $x \in X-\bar{U}$. But since F converges to x, there must exist a neighborhood V of x, such that $(V \cap O) \subset X-\bar{U}$. This then would imply that $(V \cap O) \cap \bar{U}=\phi$, contradicting the fact that F s-accumulate to x_0. Therefore X is not extremally disconnected is false.

Corollary) If X is a Hausdorff space, then X is extremally disconnected.

Corollary) Let X be a regular compact space, Then X is S-closed if and only if X is extremally disconnected.

Theorem 3) The semi-continuous surjection of a S-closed space onto any Hausdorff space is H-closed.

Proof) Let $f: X \to Y$ be a semi-continuous surjection and V an arbitrary open cover of Y. Then $\{f^{-1}(Va)\}$ is a semi-open cover of X. By hypothesis, there exists a finite subfamily such that $\cap^n_{i=1}\overline{f^{-1}(Va_i)}=X$. Notice that $\cup^n_{i=1}f^{-1}(Va_i)$ being dense in X implies

$\underline{\bigcup_{i=1}^{n} f(Va_i)} = X$. Then
$Y = f(X) = f(\bigcup_{i=1}^{n} f^{-1}(Va_i)) \subset f(\overline{\bigcup_{i=1}^{n} f^{-1}(Va_i)})$
$= \overline{\bigcup_{i=1}^{n} Va_i} = \bigcup_{i=1}^{n} \overline{Va_i}$. Therefore, Y is H-closed.

Corollary) The semi-continuous surjection of an S-closed space onto any regular space is compact.

Theorem 4) If $f : X \to Y$ is irresolute if and only if for every subset A of X, $f(\underline{A}) \subset \underline{f(A)}$.

Theorem 5) If $f : X \to Y$ is an irresolute sutjection from an S-closed space X, then Y is S-closed.

Proof) Let {Va} be a semi-open cover of Y. Then $\{f^{-1}(Va)\}$ is a semi-open cover of X and has a finite subfamily such that $\bigcup_{i=1}^{n} \overline{f^{-1}(Va_i)} = X$. Since $\bigcup_{i=1}^{n} f^{-1}(Va_i)$ is dense in X, $\bigcup_{i=1}^{n} f^{-1}(Va_i) = X$. Therefore
$Y = f(X) = f(\underline{\bigcup_{i=1}^{n} f^{-1}(Va_i)}) \subset f(\underline{\bigcup_{i=1}^{n} f^{-1}(Va_i)})$
$= \underline{\bigcup_{i=1}^{n} Va_i} \subset \bigcup_{i=1}^{n} \overline{Va_i}$. Hence Y is S-closed.

Theorem 6) The irresolute image of any S-closed Hausdorff space is closed.

Proof) Let $f : X \to Y$ be an irresolute function from an S-closed space Y. Let $y \in \overline{f(X)}$ and N(y) be the open neighborhood filterbase about y. By hypothesis, the filterbase $F = f^{-1}(N(y))$ has an s-accumulation point x. We claim that the filterbase f(F) accumulates to f(x) in the usual sense. Indeed, let V be any open set containing f(x). Then $f^{-1}(V)$ is a semi-open set containing x. and therefore for every $W \in N(y)$, $f^{-1}(W) \in F$, and
$f^{-1}(W) \cap \overline{f^{-1}(V)} \neq \phi$.
But $f^{-1}(W)^{\circ} \cap f^{-1}(V)^{\circ} \neq \phi$. Therefore
$\phi \neq f(f^{-1}(W)^{\circ} \cap f^{-1}(V)^{\circ})$
$\subset f(f^{-1}(W) \cap f^{-1}(V)) \subset W \cap V$.

Since W and V were arbitrarily chosen, we have that f(F) accumulates to f(x). But f(F) is a finer filterbase than N(y), hence N(y) accumulates to f(x). Since N(y) obviously converges to y, by the property of Hausdorff space,
$f(x) = y$. Hence $y \in f(X)$ and f(X) is closed in Y

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